

ON THE (LC) CONJECTURE

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ABSTRACT. We prove the (LC) conjecture of Hochster and Huneke in some non-trivial cases. This has several applications. Recently, Brenner and Caminata answered a numerical evidence due to Dao and Smirnov on the shape of generalized Hilbert-Kunz functions of smooth curves. As applications, we first reprove this by a short argument. Then we give a proof of second numerical evidence predicted by Dao and Smirnov on the shape of generalized Hilbert-Kunz functions of nodal curves. Thirdly, we answer a question posted by Vraciu on the (LC) property of a proposed ring. Inspiring with the (LC) property, we present a connection to the stability theory. This leads us to investigate the stability and the strong semistability of the sheaf of relations on $\{x^2, y^2, z^2\}$ over the Klein's quartic curve. This answers questions of Brenner. After presenting a connection from (LC) to the F -threshold, we answer a question posted by Huneke et al. Additional applications and examples are given.

1. INTRODUCTION

Throughout this paper $R := \bigoplus_{n \geq 0} R_n$ is a standard graded algebra over a field R_0 of prime characteristic $p > 0$, $\mathfrak{m} := \bigoplus_{n > 0} R_n$ is the irrelevant ideal and $I \triangleleft R$ is a homogeneous ideal, otherwise specializes. For each $n \in \mathbb{N}$, set $q := p^n$ and denote the n -th Frobenius power by $I^{[q]} := (x^q : x \in I)$. By $H_{\mathfrak{m}}^0(R/I^{[q]})$ we mean the elements of $R/I^{[q]}$ that annihilated by some powers of \mathfrak{m} . This *section* functor was introduced by Grothendieck. We call them *Local Cohomology* modules. By definition, there is $f(q) \in \mathbb{N}_0$ depending on q such that $\mathfrak{m}^{f(q)} H_{\mathfrak{m}}^0(R/I^{[q]}) = 0$. The (LC) conjecture claims that $f(q)$ is of linear type:

Conjecture 1.1. There is some $b \in \mathbb{N}_0$ that does not depending to q such that $\mathfrak{m}^{bq} H_{\mathfrak{m}}^0(R/I^{[q]}) = 0 \quad \forall q$.

Hochster and Huneke [25] introduced the (LC) conjecture in relation to the *localization problem*. In [3], Brenner and Monsky found a counterexample to the localization problem. In this paper we investigate the (LC) property in some non-trivial cases and present some of its applications.

The unsolved question in this area is whether *weakly F -regular* rings are *F -regular*. By a result of Huneke, the (LC) property in dimension one answers this question affirmatively. We now explain how the (LC) condition arises from Hilbert-Kunz theory. By $f_{gHK}^{R/I}(n)$ we mean the length of $H_{\mathfrak{m}}^0(R/I^{[q]})$ as an R -module and by $e_{gHK}(R/I)$ we mean $\lim_{n \rightarrow \infty} \frac{f_{gHK}^{R/I}(n)}{p^{n \dim R}}$, if the limit exists and put ∞ otherwise. Following Dao and Smirnov [15], we call them the *generalized Hilbert-Kunz function* and the generalized Hilbert-Kunz multiplicity, respectively. They proved such a limit exists under some conditions. Very recently, Vraciu [49] observed that such a limit exists for a subclass of rings that satisfy in the (LC) property. Our motivation comes from a paper of Brenner on the *irrational* possibility of Hilbert-Kunz multiplicity. He proved this by observing that

$$e_{gHK}(R/(a, b)) \notin \mathbb{Q} \quad (\star),$$

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where R is the coordinate ring of a $K3$ -surface in \mathbb{P}^3 , see [8]. Hilbert-Kunz multiplicity introduced by Kunz [34], and Monsky proved that such a limit exists [40].

The (LC) conjecture is true in 2-dimensional normal rings by Vraciu [50]. In the graded rings and in the case $\dim R/I = 1$ the (LC) checked by Vraciu [50] and Huneke [26]. Also, we cite the work of Katzman as another related source [32].

The organization of this paper is as follows. In Section 2 we summarize some known results that we need. Section 3 introduces the (LC) property with respect to a family of ideals. The focus here is limited to the Frobenius powers. However, we catch a glimpse to certain families of ideals. For any $X \subset \operatorname{Spec}(R)$, set $X^i := \{\mathfrak{p} \in X \mid \dim R/\mathfrak{p} \geq i\}$. Section 4 is devoted to show:

Theorem 1.2. *Let R be a standard graded Cohen-Macaulay over a field R_0 of prime characteristic $p > 0$ and let $I \triangleleft R$ be homogeneous. Suppose one of the following holds:*

- (i) *The ring is normal and $\dim R < 4$.*
- (ii) *The ring is normal and $\operatorname{Proj}(\frac{R}{I})^2$ is regular.*
- (iii) *$\operatorname{p.dim}(R/I) < \infty$.*

Then $\mathfrak{m}^{bq} H_{\mathfrak{m}}^0(R/I^{[q]}) = 0$ for some b that does not depending to q .

This motivates to check the (LC) property over rings with *quotient singularity*, see Remark 4.6. One may recover this by Discussion 4.7: The (LC) property holds for rings of *finite F -representation type*. In Section 5 we apply Theorem 1.2 to deduce the following which was asked by Vraciu [49, Page 3]:

Corollary 1.3. *The R -module $R/(a, b)$ in (\star) satisfies in the (LC) condition.*

This implies that $f_{gHK}^{R/I}$ is a linear combination with integer coefficients of Hilbert-Kunz functions of \mathfrak{m} -primary ideals. By the help of this and (\star) Vraciu obtains a more direct \mathfrak{m} -primary ideals with irrational Hilbert-Kunz multiplicities. In Section 6, we first reprove the main result of [7] (at the level of ideals) by a short argument. This was asked in [15] and regards as an application of the (LC) condition:

Corollary 1.4. *Over 2-dimensional normal graded domains over a field F of prime characteristic, one has:*

- i) $f_{gHK}^{R/I}(n) = e_{gHK}(R/I)q^2 + \gamma(q)$, where $e_{gHK}(R/I) \in \mathbb{Q}$ and $\gamma(q)$ is a bounded function,
- ii) if $F = \overline{\mathbb{F}}_p$, then $\gamma(q)$ is an eventually periodic function.

In order to present the next application, we look at the non-normal ring $R := \frac{k[[x, y, t]]}{(x^3 + txy + y^3)}$ and set $M := R/(x, y)$. For e.g. $k = \mathbb{F}_{11}$ recall from [15, Example 6.3(3)] the following numerical evidence: " $f_{gHK}^M(q) = \frac{q^2 + 2q - \gamma(q)}{3}$ and the formula seems to depend on whether $q \equiv 1 \pmod{3}$." We apply the (LC) condition to prove this numerical evidence:

Corollary 1.5. *Suppose R is the coordinate ring of a nodal cubic plane projective curve over an algebraically closed field of prime characteristic p and $I \triangleleft R$ is graded. Then $f_{gHK}^{R/I}(q) = \mu q^2 + aq - r$, where r is an integer that depends on $q \pmod{3}$.*

The next subject of Section 6 is Theorem 6.7: As an application of (LC), we quickly derive $e_{gHK}(-) \in \mathbb{Q}$ for the coordinate ring of singular plane projective curves. In the sequel we need the following quantities:

Definition 1.6. For an ideal I of a ring R , set:

- i) $b(I) := \inf\{b \in \mathbb{N}_0 : \mathfrak{m}^{bq} H_{\mathfrak{m}}^0(R/I^{[q]}) = 0 \quad \forall q\}$,

- ii) $c(I) := \inf\{b \in \mathbb{N}_0 : \mathfrak{m}^{bq} H_{\mathfrak{m}}^0(R/I^{[q]}) = 0 \quad \forall q \gg 0\},$
- iii) $d(I) := \inf\{b \in \mathbb{N}_0 : \mathfrak{m}^{bn} H_{\mathfrak{m}}^0(R/I^n) = 0 \quad \forall n \gg 0\}.$

Suppose $I \triangleleft R$ is generated by linear forms and \mathfrak{m} -primary. Let us recall from [16, Proposition 0.5] that $d(I) \leq 2$. In Section 7 first we show:

Corollary 1.7. *Let $I = (f, g)$ be a homogeneous 2-generated prime ideal of the ring R in Corollary 1.4 of height one and let $s \in \mathfrak{m}^c \setminus I$ be homogeneous, where $c := b(I)$ is the (LC) exponent. Then at least one of the syzygy bundles $\{\text{Syz}(f, g, s), \text{Syz}(f, g, s^2), \text{Syz}(f, g, s^4)\}$ is not strongly semistable.*

Let \mathcal{C} be a degree four plane curve. Brenner proved that $\mathcal{V} := \text{Syz}_{\mathcal{C}}(x^2, y^2, z^2)$ is semistable, see [9, Lemma 7.1] and he posted two questions on the stability and the strongly semistability of \mathcal{V} over *Klein's quartic* $zx^3 + xy^3 + yz^3 = 0$. We note that *Mumford's stability* has essential applications to Hilbert-Kunz theory. This contribution was made by Brenner and Trivedi. Reversely, we answer Brenner's questions:

Example 1.8. Let $R := \overline{\mathbb{F}}_2[x, y, z]/(zx^3 + xy^3 + yz^3)$ and \mathcal{C} be the corresponding curve. The following holds: i) \mathcal{V} is stable, ii) \mathcal{V} is not strongly semistable.

In Section 8 and when I is \mathfrak{m} -primary, we give a connection from $c(I)$ to $c^I(\mathfrak{m})$ the *F-threshold* of I with respect to \mathfrak{m} : $c(I) - 1 \leq c^I(\mathfrak{m}) \leq c(I)$, see Observation 8.1 for a more precise statement. Let us recall the following application of *F-threshold* from [27, Example 3.4]: "Let (R, \mathfrak{m}) be a regular local ring of characteristic $p > 0$ with $\dim(R) = d$ and let J be an ideal of R generated by a full system of parameters. We define a to be the maximal integer n such that $\mathfrak{m}^n \not\subseteq J$. Then $\mathfrak{m}^s \subseteq \overline{J}$ if and only if $s \geq \frac{a}{d} + 1$." Then, Huneke-Mustata-Takagi-Watanabe send two questions ([27, Questions 3.5]): Does this statement hold in a more general setting? Can we replace "regular" by "Cohen-Macaulay"?

Example 1.9. The above questions have negative answers.

In Section 9 we connect $d(\sim)$ to the *Waldschmidt constant*. In Section 10 we focus on $R := \overline{\mathbb{F}}_p[X_1, \dots, X_m]$ and on a graded ideal $I \triangleleft R$. We show in Proposition 10.4 that $f_{g_{HK}}^{R/I}(n) = e_{g_{HK}}(R/I)q^m(\mathfrak{h})$. Also, $e_{g_{HK}}(R/I)$ realizes as a length of a module. Compare this with the irrational possibility of $\lim_{n \rightarrow \infty} \frac{H_{\mathfrak{m}}^0(R/I^n)}{n \dim R}$, see [14]. One may regards (\mathfrak{h}) as a Frobenius version of a question of Herzog, see Question 10.1. We present a reformulation of Question 10.1 by the help of an algorithm, see Corollary 10.2.

Section 11 goes on to investigate $b(I)$ and $c(I)$. Let Γ be a family of two-generated ideals of fixed degrees in the coordinate ring of smooth plane projective curve. We apply in Remark 11.5 a computational method of Brenner to give a bound on $\{b(I) : I \in \Gamma\}$. However, we give an example such that $\sup\{b(I^n)\} = \infty$ as n varies. In a similar vein:

Example 1.10. There is no polynomial function as F such that $F(e_{g_{HK}}(R/I), e_{g_{HK}}(R/J)) = e_{g_{HK}}(R/IJ)$ even if I and J projectively have the same *closure operation*.

This demonstrates a different behavior between $e_{g_{HK}}(\sim)$ and $e_{HK}(\sim)$ among several similarities. This raises through a question of Brenner and Caminata [7]. The (LC) exponents live in the shadow of a degree data coming from the ideal and the ring, at least in the examples presented in Section 11.

In our final section we collect few remarks in the local situation.

2. PRELIMINARIES

All rings in this paper are commutative, Noetherian and of prime characteristic p . In the next subsections, we set up the basic foundation for the paper. In particular, let A be a ring with an ideal \mathfrak{a} with a generating set $\underline{a} := a_1, \dots, a_r$. By $H_{\underline{a}}^i(M)$ we mean the i -th cohomology of Čech complex of a module M with respect to \underline{a} . This is independent of the choose of the generating set. For simplicity, we denote it by $H_{\mathfrak{a}}^i(M)$. By \mathbb{N}_0 we mean $\mathbb{N} \cup \{0\}$. The grade of \mathfrak{a} on M is defined by

$$\text{grade}_A(\mathfrak{a}, M) := \inf\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(M) \neq 0\}.$$

We use $\text{depth}(M)$, when we deal with the maximal ideal of $*$ -local rings. By *Grothendieck's vanishing theorem*, we have $H_{\mathfrak{a}}^i(M) = 0$ for all $i > \dim M$.

Definition 2.1. Let I be an ideal of a (graded) local ring (R, \mathfrak{m}) . There is an integer $t \in \mathbb{N}_0$ such that $\bigcup (I : \mathfrak{m}^m) = (I : \mathfrak{m}^t)$. Thus $H_{\mathfrak{m}}^0(R/I) = \frac{(I : \mathfrak{m}^t)}{I}$. The ideal $I^{\text{sat}} := (I : \mathfrak{m}^t)$ is called the saturation of I .

Fact 2.2. The saturation of I computed as the intersection of all primary to nonmaximal prime ideals associated to I .

Denote the n -the symbolic power of I by $I^{(n)} := \bigcap_{\mathfrak{p} \in \text{Ass}(I)} (I^n R_{\mathfrak{p}} \cap R)$.

Fact 2.3. Suppose \mathfrak{p} is a prime ideal of dimension one. Recall that $\mathfrak{p}^{(n)}/\mathfrak{p}^n = H_{\mathfrak{m}}^0(R/\mathfrak{p}^n)$.

The assignment $a \mapsto a^p$ defines a ring homomorphism $F : A \rightarrow A$. By $F^n(A)$, we mean A as a group equipped with left and right scalar multiplication from A given by

$$a \cdot r \star b = ab^{p^n} r, \text{ where } a, b \in A \text{ and } r \in F^n(A),$$

Now we recall the *Peskine-Szpiro* functor. For an A -module M , set $F^n(M) := F^n(A) \otimes M$. The left R -module structure of $F^n(A)$ endows $F^n(M)$ with a left R -module structure such that $c(a \otimes x) = (ca) \otimes x$. For an R -linear map $\varphi : M \rightarrow N$ one considers $F^n(\varphi)$ to be the R -linear map $\text{id}_{F^n(A)} \otimes \varphi$. The assignment $a \otimes (r + \mathfrak{a}) \mapsto ar^{p^n} + \mathfrak{a}^{[p^n]}$ defines an isomorphism $\varphi : F^n(A/\mathfrak{a}) \rightarrow A/\mathfrak{a}^{[p^n]}$.

Remark 2.4. Suppose A is graded. To make φ degree-preserving, define a new grading on $A/\mathfrak{a}^{[p^n]}$ given by $\deg^{\text{new}}(r + \mathfrak{a}^{[p^n]}) = \frac{\deg(r)}{p^n}$. Having this grading (resp. the usual grading) in mind, we denote the terms of (\sim) of degree greater or equal than ℓ by $(\sim)_{\geq \ell}$ (resp. $(\sim)_{\geq \ell}$). Then we have

$$H_{\mathfrak{m}}^0(F^n(A/\mathfrak{a}))_{\geq b} = 0 \implies H_{\mathfrak{m}}^0(A/\mathfrak{a}^{[q]})_{\geq bq} = 0.$$

Fact 2.5. Over regular rings the Frobenius map is flat, see [11, Corollary 8.2.8].

Fact 2.6. For any multiplicative closed set S , $S^{-1}F^n(M) \simeq F^n(S^{-1}M)$, see [11, Proposition 8.2.5].

The set A° denotes the elements in A which are not in any minimal prime ideals. Also, we note that $\mathfrak{a}^{[p^n]} = (a_1^{p^n}, \dots, a_r^{p^n})$ is independent of the choose of the generating set. Recall from [25] that the tight closure of \mathfrak{a} is: $\mathfrak{a}^* := \{x \in A : \exists c \in A^\circ \text{ s.t. } cx^q \in \mathfrak{a}^{[q]} \ \forall q > 0\}$. A ring in which all ideals are tightly closed is called weakly F -regular. A ring is called F -regular, if all of its localizations are F -regular. A is called strongly F -regular if for every $c \in A^\circ$ the homomorphism $A \rightarrow A^{1/q}$ sending $1 \mapsto c^{1/q}$ splits.

Recall that by $f_{gHK}^{A/\mathfrak{a}}(n)$ we mean the length of $H_{\mathfrak{m}}^0(A/\mathfrak{a}^{[q]})$ as an R -module. Now, if \mathfrak{a} is of finite colength, then we are in the situation of the classical Hilbert-Kunz multiplicity and we use $f_{HK}^{A/\mathfrak{a}}$ (resp. $e_{HK}(A/\mathfrak{a})$) instead of $f_{gHK}^{A/\mathfrak{a}}$ (resp. $e_{gHK}(A/\mathfrak{a})$). The limit $e_{HK}(A/\mathfrak{a}) := \lim_{n \rightarrow \infty} \frac{f_{HK}^{A/\mathfrak{a}}(n)}{p^n \dim A}$ exists by Monsky [40].

3. THE (LC) PROPERTY

Let $\{I_n\}$ be a family of ideals. For a fixed I , the following are the main examples: i) $I_n := I^n$; ii) $I_n := I^{[q]}$; and iii) $I_n := (I^{[q]})^*$. The next item appears in the base changing: Suppose S is any ring with a family of ideals $\{J_n\}$ and suppose there is a map $S \rightarrow R$. Then $I_n := J_n R$ is a family of ideals.

We say (LC) holds for $\{I_n\}$, if there is $b \in \mathbb{N}_0$, does not depending to q such that $\mathfrak{m}^{bq} H_{\mathfrak{m}}^0(R/I_n) = 0$ for all q . This imposes strong condition on the family:

Example 3.1. Set $I_n := \mathfrak{m}^{q^n}$. Then $\mathfrak{m}^{f(q)} H_{\mathfrak{m}}^0(R/I_n) \neq 0$ for any polynomial f .

The following example of Kollar [33, Example 1.4] has a role in the effective nullstellensatz.

Example 3.2. Let $A := \mathbb{C}[x, y, z, s]$ and let $\mathfrak{a}_n := (x^2 - y^{2n+1}, z^2, xz, y^n z, s)$. The primary decomposition of \mathfrak{a}_n is $\mathfrak{a}_n = (x^2 - y^{2n+1}, z, s) \cap (x^2 - y^{2n+1}, z, s, x, y^n)$. Set $\mathfrak{m} := (x, y, z, s)$. The \mathfrak{m} -primary component of \mathfrak{a}_n is $(x^2 - y^{2n+1}, z, s, x, y^n)$. By, $H_{\mathfrak{m}}^0(A/\mathfrak{a}_n) \simeq \frac{\mathfrak{a}_n^{sat}}{\mathfrak{a}_n} \simeq \frac{(x^2 - y^{2n+1}, z, s)}{\mathfrak{a}_n}$. So, $\mathfrak{m}^n H_{\mathfrak{m}}^0(A/\mathfrak{a}_n) = 0$.

In this paper we are mainly interested in the Frobenius power of an ideal and by the (LC) we mean the (LC) with respect to the Frobenius powers. We need the following trick of Hochster and Huneke:

Fact 3.3. Let A be a noetherian ring, with a maximal ideal \mathfrak{m} and \mathfrak{a} is an ideal of dimension one satisfies in $\mathfrak{m}^{bq} H_{\mathfrak{m}}^0(A/\mathfrak{a}^{[q]}) = 0$. Then \mathfrak{a}^* commutes with the localization with respect to $\{a^n\}$ where $a \in A$. Indeed, take $x \in (\mathfrak{a} A_a)^*$. After replacing x by $f^{q_0} x$ we assume that $x \in A$. By the same trick and by Definition 2.1, there is $c \in A^\circ$ such that $cx^q \in \mathfrak{a}^{[q]} A_a \forall q$. There is $f(q)$ depending on q such that $ca^{f(q)} x^q \in \mathfrak{a}^{[q]}$. This says

$$cx^q \in H_{\mathfrak{a}}^0(A/\mathfrak{a}^{[q]}) = H_{\mathfrak{a} + \mathfrak{a}^{[q]}}^0(A/\mathfrak{a}^{[q]}) = H_{\text{rad}(\mathfrak{a} + \mathfrak{a}^{[q]})}^0(A/\mathfrak{a}^{[q]}) = H_{\mathfrak{m}}^0(A/\mathfrak{a}^{[q]}).$$

By the (LC) condition there is b such that $c(a^b x)^q \in \mathfrak{a}^{[q]}$. Thus, $a^b x \in \mathfrak{a}^*$ and so $x \in \mathfrak{a}^* A_a$. The other side inclusion is trivial.

Remark 3.4. Having the (LC) condition for ideals of dimension one, then being weakly F -regular implies F -regular, see the discussion after [26, Corollary 3.2]. We recall from [37] that weak F -regularity and strong F -regularity agree for standard \mathbb{N} -graded rings which are of F -finite type over a field.

Here is a comment on a graded algebra with a base ring which is not a field.

Remark 3.5. Let $R := \frac{\mathbb{F}_2[x, y, z, t]}{(z^4 + xy z^2 + z(x^3 + y^3) + (t + t^2)x^2 y^2)}$ and $I := (x^4, y^4, z^4)$. The following holds.

- i) (See [3, Main result]) Let $S := \mathbb{F}_2[t] \setminus \{0\}$. Then $(S^{-1}I)^* \neq S^{-1}(I^*)$.
- ii) We note that R is a 3-dimensional Cohen-Macaulay integral domain and $\dim R/I = \dim R - 1$. Under the assignments $\deg(t) = 0$ and $\deg(x) = \deg(y) = \deg(z) = 1$, R is graded with the irrelevant ideal $\mathfrak{p} := (x, y, z)$. However, $R_0 := \mathbb{F}_2[t]$ is not a field.
- iii) There is $b \leq 660$ such that $\mathfrak{p}^{bq} H_{\mathfrak{p}}^0(R/I^{[p^n]}) = 0$. First note that $H_{\mathfrak{p}}^0(R/I^{[p^n]}) = R/I^{[p^n]}$ and $\mathfrak{p}^{10} \subset I$. Indeed, suppose $x^i y^j z^k \in \mathfrak{p}$ is a monomial with $i + j + k = 10$ and $i, j < 4$. Then $k \geq 4$ and so $x^i y^j z^k \in I$. The set

$$\{x^i y^j z^k : i + j + k = 10\} = \{x^{10}\} \cup \{x^9 y, x^9 z\} \cup \{x^8 y^2, x^8 z^2, x^8 y z\} \cup \dots \cup \{y^{10}, y^9 z, \dots, z^{10}\}$$

generates \mathfrak{p}^{10} . Its cardinality is $1 + 2 + 3 + \dots + 11 = 66$. Set $a := 10 \times 66$. Then $\mathfrak{p}^{aq} \subseteq (\mathfrak{p}^{10})^{[q]} \subseteq I^{[q]}$ and so $\mathfrak{p}^{aq} H_{\mathfrak{p}}^0(R/I^{[p^n]}) = 0$.

- iv) Let $0 \neq a \in R$ be homogeneous of positive degree. Then $(IR_a)^* = (I^*)_a$. The claim follows by the above item and the proof of Fact 3.3 (this may follows directly).

v) Computing $H_{f+\mathfrak{p}}^0(R/I^{[p^n]})$ has a significant importance, where $0 \neq f \in \mathbb{F}_2[t]$. We revisit this in the near future, as another perspective of the (LC)-condition.

The following plays an essential role in this paper.

Lemma 3.6. *Let $\{I_n\}$ be a family of ideals of dimension one that satisfies in the (LC) property, i.e., $\mathfrak{m}^{aq}H_{\mathfrak{m}}^0(R/I_n) = 0$ for some a . Suppose that $\Gamma := \{\mathfrak{p} : \mathfrak{p} \in \text{Ass}(R/I_n)\} \setminus \{\mathfrak{m}\}$ satisfies in the prime avoidances. Let $x \in \mathfrak{m}^a \setminus \bigcup_{\mathfrak{p} \in \Gamma} \mathfrak{p}$. Then $\ell(H_{\mathfrak{m}}^0(R/I_n)) = 2\ell(\frac{R}{I_n+(x^n)}) - \ell(\frac{R}{I_n+(x^{2n})})$.*

Proof. This is a routine modification of [49, Proposition 2.4] where the claim proved for the Frobenius power. We left the details to the reader. \square

4. PROOF OF THEOREM 1.2

Lemma 4.1. *(See e.g. [50, Proposition 2]) Suppose $\dim R \leq 1$ and M is graded. Then $\mathfrak{m}^{bq}H_{\mathfrak{m}}^0(F^n(M)) = 0$ for some b that does not depending to q .*

Lemma 4.2. *Suppose one of the following holds:*

- i) *The ring R is normal, 2-dimensional and I is an ideal,*
- ii) *The ring R is finitely generated, positively graded algebra over a field k , and I a homogeneous ideal. Suppose $\text{ht}(I) = \dim R - 1$.*

Then $\mathfrak{m}^{bq}H_{\mathfrak{m}}^0(R/I^{[q]}) = 0$ for some b that does not depending to q where \mathfrak{m} is the maximal ideal.

Proof. The former is in [50, Proposition 1] and the later is in [50, Theorem 1] and [26]. \square

Let $R = \bigoplus_{n \geq 0} R_n$ be a standard graded ring, $\mathfrak{m} := \bigoplus_{n > 0} R_n$ the irrelevant ideal and $L = \bigoplus_{n \in \mathbb{Z}} L_n$ a graded R -module. Set $\text{end}(L) := \text{Supp}\{n : L_n \neq 0\}$. Clearly, $\text{end}(\bigoplus_{i \in I} L(-\ell_i)) = \max\{\ell_i\} + \text{end}(L)$ and $\text{end}(L_1 \oplus L_2) = \max\{\text{end}(L_1), \text{end}(L_2)\}$. Let $0 \neq M$ be a finitely generated and graded R -module. Then $H_{\mathfrak{m}}^i(M)$ is \mathbb{Z} -graded. The well-known point is that $\text{end}(H_{\mathfrak{m}}^i(M)) < \infty$. Let $\ell \in \mathbb{Z}$. Recall that by $L_{\geq \ell}$ we mean $\bigoplus_{i \geq \ell} L_i$.

Lemma 4.3. *Let R be a standard graded ring over a field of characteristic $p > 0$ and M be finitely generated and graded. If $\text{end}(H_{\mathfrak{m}}^0(F^n(M))) \leq aq + c$ for some a and c not depending to q , then M satisfies in the (LC) property.*

Proof. Write $c = dq + e$ where $0 \leq e \leq q - 1$ and $d \in \mathbb{N}_0$. Set $f := a + (d + 1)$. Then

$$\text{end}(H_{\mathfrak{m}}^0(F^n(M))) < fq \quad (\star)$$

for some f independent of q . We have $\mathfrak{m}(H_{\mathfrak{m}}^0(F^n(M)))_i \subseteq H_{\mathfrak{m}}^0(F^n(M))_{\geq i+1}$. As M is finitely generated and R is positively graded, there is ℓ_0 such that $M_i = 0$ for all $i < \ell_0$. If

$$\bigoplus_i R(-\beta_{1i}) \xrightarrow{(a_{ij})} \bigoplus_i R(-\beta_{0i}) \longrightarrow M \longrightarrow 0.$$

is the graded presenting sequence of M , then

$$\bigoplus_i R(-q\beta_{1i}) \xrightarrow{(a_{ij}^q)} \bigoplus_i R(-\beta_{0i}q) \longrightarrow F^n(M) \longrightarrow 0$$

is the graded presenting sequence of $F^n(M)$, because the tensor product is right exact and $F^n(a_{ij}) = (a_{ij}^q)$. This yields that $H_{\mathfrak{m}}^0(F^n(M)) \subseteq F^n(M)$ concentrated in degrees greater than $\ell_0 q$. Set $b := f + \lfloor \ell_0 \rfloor$. Therefore

$$\mathfrak{m}^{bq}H_{\mathfrak{m}}^0(F^n(M)) \subseteq H_{\mathfrak{m}}^0(F^n(M))_{\geq fq} \stackrel{(\star)}{=} 0$$

for some b that does not depending to q . \square

Set $b_0(L) := \inf\{\ell : \bigoplus_i L_{i \leq \ell} R = L\}$. We will utilize the following result.

Lemma 4.4. ([35, Lemma 3.2]) *Let R be a standard graded ring over a field of characteristic $p > 0$ and M a finitely generated graded R -module. Let*

$$\cdots \longrightarrow C_d \longrightarrow \cdots \xrightarrow{\pi} C_0 \longrightarrow 0$$

be a graded complex with $\text{coker}(\pi) = M$. Suppose the following assertions hold:

- (1) $\text{depth } C_j \geq i + j + 1$ for all $0 \leq j \leq \dim R - i - 1$, and
- (2) $\dim H_j(C_\bullet) \leq j + i$ for all $j \geq 1$.

Then $\text{end}(H_{\mathfrak{m}}^i(M)) \leq b_0(C_{\dim R - i}) + \text{end}(H_{\mathfrak{m}}^{\dim R}(R))$.

Corollary 4.5. *Let R be graded Cohen-Macaulay and I a homogeneous ideal of finite projective dimension. Then $\mathfrak{m}^{bq} H_{\mathfrak{m}}^0(R/I^{[q]}) = 0$ for some b that does not depending to q .*

Proof. Denote the minimal free resolution of R/I by F_\bullet . In view of [44, Theorem 1.13], $F^n(F_\bullet)$ is the minimal free resolution of $R/I^{[q]}$, i.e., Lemma 4.4(2) satisfied. The Cohen-Macaulay property implies Lemma 4.4(1). So, we are in the situation to apply Lemma 4.4. That is $\text{end}(H_{\mathfrak{m}}^0(R/I^{[q]})) \leq aq + c$ for some a and c not depending to q . Now Lemma 4.3 presents the desired claim. \square

Remark 4.6. Let R be graded (not necessarily standard) with only quotient singularity. Then for any graded ideal I , one has $\mathfrak{m}^{bq} H_{\mathfrak{m}}^0(R/I^{[p^n]}) = 0$ for some b that does not depending to q .

Let us recall the concept of quotient singularity. This means that there is a finite extension $R \subset S$ of graded rings such that $R = S^G := \{s \in S : g(s) = s \ \forall g \in G\}$, where S is a standard graded regular ring and G a finite group of order prime to the characteristic. The trace map divided by the group order shows that R is a direct summand of S . It may be worth to recover Remark 4.6 by a more general technic:

Discussion 4.7. (i) Recall from [46] that R has finite F -representation type if there are finitely generated \mathbb{Q} -graded R -modules $M_1 \dots, M_s$ such that for any q , there exist nonnegative integers m_{qi} and rational numbers $\ell_{i,j}^{m_{qi}}$ such that $F^n(R) \simeq \bigoplus_{1 \leq i \leq s} \bigoplus_{1 \leq j \leq m_{qi}} M_i(\ell_{i,j}^{m_{qi}})$. This is well-known that the sequence $\{\ell_{i,j}^{m_{qi}}\}$ is bounded from below.

(ii) The following are of finite F -representation type: 1) $R_1 := \mathbb{F}_p[X_1, \dots, X_n]$, 2) $R_2 := R_1/I$ where I is a monomial ideal, 3) direct-summand of R_2 .

(iii) Let R be a standard graded ring and of finite F -representation type over an F -finite field R_0 and let $I \triangleleft R$ be homogeneous. Then $\mathfrak{m}^{bq} H_{\mathfrak{m}}^0(R/I^{[q]}) = 0$ for some b that does not depending to q . Indeed, we do with the similar lines throughout [30, Theorem 7.6]:

$$H_{\mathfrak{m}}^0(F^n(R/I)) = H_{\mathfrak{m}}^0(F^n(R) \otimes R/I) = \bigoplus H_{\mathfrak{m}}^0(M_j/IM_j)(\ell_{i,j}^{m_{qi}})$$

Having $c := \inf_{i,j,m_{qi}} \{\ell_{i,j}^{m_{qi}}\}$ from part (i),

$$\text{end}(H_{\mathfrak{m}}^0(F^n(R/I))) \leq \max_{1 \leq j \leq s} \{\text{end}(H_{\mathfrak{m}}^0(\frac{M_j}{IM_j}))\} - c < \infty.$$

Set $b := \max_{1 \leq j \leq s} \{\text{end}(H_{\mathfrak{m}}^0(\frac{M_j}{IM_j}))\} - c < \infty$. In view of Remark 2.4, $H_{\mathfrak{m}}^0(R/I^{[q]})_{\geq bq} = 0$ and deduce from Lemma 4.3 that $\mathfrak{m}^{bq} H_{\mathfrak{m}}^0(R/I^{[q]}) = 0$.

Notation 4.8. By $\mathcal{R}(n)$ and $\mathcal{S}(n)$ we denote the Serre's conditions. The property $\mathcal{R}(n)$ defined as follows: If $\mathfrak{p} \in \text{Spec}(R)$ and $\text{ht}(\mathfrak{p}) \leq n$, then $R_{\mathfrak{p}}$ is regular, and $\mathcal{S}(n)$ defines by $\text{depth}(R_{\mathfrak{p}}) \geq \min\{n, \text{ht}(\mathfrak{p})\}$.

The following is a higher dimensional version of [7, Lemma 3.7].

Proposition 4.9. *Let R be a graded domain of dimension $d > 1$ satisfies in $\mathcal{S}(2)$, $J \triangleleft R$ and $f \in R$ be homogeneous. Then $f_{gHK}^{R/J} = f_{gHK}^{R/fJ}$. In particular, $e_{gHK}(R/J) = e_{gHK}(R/fJ)$.*

Proof. As R is $\mathcal{S}(2)$, one has $\text{depth}(R_{\mathfrak{m}}) \geq 2$ and by [11, Proposition 1.5.15(e)], $\text{grade}(\mathfrak{m}, R) \geq 2$. Look at $0 \rightarrow J^{[p^n]} \rightarrow R \rightarrow R/J^{[p^n]} \rightarrow 0$. It induces

$$0 \simeq H_{\mathfrak{m}}^0(R) \rightarrow H_{\mathfrak{m}}^0(R/J^{[p^n]}) \rightarrow H_{\mathfrak{m}}^1(J^{[p^n]}) \rightarrow H_{\mathfrak{m}}^1(R) \simeq 0.$$

So, $H_{\mathfrak{m}}^0(R/J^{[p^n]}) \simeq H_{\mathfrak{m}}^1(J^{[p^n]})$ (*). Similarly, $H_{\mathfrak{m}}^0\left(\frac{R}{(fJ)^{[p^n]}}\right) \simeq H_{\mathfrak{m}}^1((fJ)^{[p^n]})$. As R is an integral domain, multiplication by f^{p^n} gives $J^{[p^n]} \simeq (fJ)^{[p^n]}$. Therefore,

$$H_{\mathfrak{m}}^0(R/J^{[p^n]}) \simeq H_{\mathfrak{m}}^1(J^{[p^n]}) \simeq H_{\mathfrak{m}}^1((fJ)^{[p^n]}) \simeq H_{\mathfrak{m}}^0(R/(fJ)^{[p^n]}).$$

This is the desired claim. \square

Lemma 4.10. *Let R be a graded $\mathcal{S}(2)$ ring over a field of characteristic $p > 0$ and $I \triangleleft R$ be graded. Then $H_{\mathfrak{m}}^0(R/I^{[q]}) \simeq H_{\mathfrak{m}}^1(I^{[q]})$ as graded modules.*

Proof. This is in the Proposition 4.9(*). \square

Serre's criterion of normality over Noetherian rings says that a ring is normal if and only if it has the properties $\mathcal{R}(1)$ and $\mathcal{S}(2)$.

Lemma 4.11. *Let R be a normal standard graded over a field of characteristic $p > 0$ and $I \triangleleft R$ be graded. Let C_{\bullet} be deleted graded free resolution of R/I and let $k > 0$. Then $\dim(H_k(F^n(C_{\bullet}))) \leq \dim R - 2$.*

Proof. In order to show this we prove $H_k(F^n(C_{\bullet}))_{\mathfrak{p}} = 0 \ \forall \mathfrak{p}$ with $\dim R/\mathfrak{p} \geq \dim R - 1$. As the ring is catenary,

$$\text{ht}(\mathfrak{p}) \leq 1 \iff \dim R/\mathfrak{p} \geq \dim R - 1.$$

By Serre's criterion, for any prime ideal \mathfrak{p} with $\text{ht}(\mathfrak{p}) \leq 1$, $R_{\mathfrak{p}}$ is regular. By Fact 2.5, the Frobenius map is flat over $R_{\mathfrak{p}}$. Also, recall that Frobenius map commutes with the localization, as Fact 2.6 dedicates. So, $H_k(F^n(C_{\bullet}))_{\mathfrak{p}} = 0$ for all \mathfrak{p} with $\dim R/\mathfrak{p} \geq \dim R - 1$. \square

Lemma 4.12. *If $L \rightarrow M \rightarrow K$ be exact sequence of graded module (not necessarily finitely generated) such that $\text{end}(L) \leq O(p^n)$ and $\text{end}(K) \leq O(p^n)$, then $\text{end}(M) \leq O(p^n)$.*

Proof. This easily reduces to the situation of short exact sequences. In this case the proof follows by definition. \square

For any $X \subset \text{Spec}(R)$, recall that $X^i := \{\mathfrak{p} \in X \mid \dim R/\mathfrak{p} \geq i\}$.

Lemma 4.13. *Suppose $\text{Proj}(\frac{R}{I})^{i+1}$ is regular. Then $\dim(\text{Tor}_j^R(F^n(R), \frac{R}{I})) \leq i$.*

Proof. Look at the graded free resolution of R/I :

$$\cdots \longrightarrow \bigoplus_i R(-\beta_{di}) \xrightarrow{\varphi_d} \cdots \longrightarrow \bigoplus_i R(-\beta_{0i}) \longrightarrow 0.$$

Apply $F^n(-)$, this induces the following complex of graded modules and graded homomorphisms

$$\cdots \longrightarrow \bigoplus_i R(-\beta_{di}q) \xrightarrow{F^n(\varphi_d)} \cdots \longrightarrow \bigoplus_i R(-\beta_{0i}q) \longrightarrow 0.$$

Its j^{th} homology is $\text{Tor}_j^R(F^n(R), \frac{R}{I})$. In particular, $\text{Tor}_j^R(F^n(R), \frac{R}{I})$ is graded. The minimal prime ideals of $\text{Supp}(\text{Tor}_j^R(F^n(R), \frac{R}{I}))$ consists of homogeneous ideals. Let \mathfrak{p} be a minimal prime ideal in $\text{Supp}(\text{Tor}_j^R(F^n(R), \frac{R}{I}))$. Suppose $\dim R/\mathfrak{p} \geq i+1$. Then $\mathfrak{p} \in \text{Proj}(\frac{R}{I})^{i+1}$. By Fact 2.6, we have

$$\text{Tor}_j^R(F^n(R), \frac{R}{I})_{\mathfrak{p}} \simeq \text{Tor}_j^{R_{\mathfrak{p}}}(F^n(R)_{\mathfrak{p}}, \frac{R_{\mathfrak{p}}}{IR_{\mathfrak{p}}}) \simeq \text{Tor}_j^{R_{\mathfrak{p}}}(F^n(R_{\mathfrak{p}}), \frac{R_{\mathfrak{p}}}{IR_{\mathfrak{p}}}).$$

Keep in mind that $R_{\mathfrak{p}}$ is regular. By Fact 2.5, $\text{Tor}_j^{R_{\mathfrak{p}}}(F^n(R_{\mathfrak{p}}), \frac{R_{\mathfrak{p}}}{IR_{\mathfrak{p}}}) = 0$ for all $j > 0$. This contradiction shows that $\dim(\text{Tor}_j^R(F^n(R), \frac{R}{I})) \leq i$. \square

Lemma 4.14. *Frobenius power commutes with the localization.*

Proof. This is straightforward and we leave it to the reader. \square

Lemma 4.15. *Let R be a regular ring with an ideal I . Then $F^n(I) \simeq I^{[q]}$ for any n .*

Proof. By Remark 2.4 $F^n(R/I) \simeq R/I^{[p^n]}$. In view of Fact 2.5, $F^n(-)$ is exact. Keep in mind $F^n(R) = R$. We apply $F^n(-)$ to the short exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ to arrives to the exact sequence: $0 \rightarrow F^n(I) \rightarrow R \rightarrow R/I^{[q]} \rightarrow 0$. So, $F^n(I) \simeq I^{[q]}$. \square

Proof of Theorem 1.2. Set $d := \dim R$. As R is Cohen-Macaulay and standard graded over a field, we have $H_{\mathfrak{m}}^i(R) = 0$ for all $0 \leq i \leq d-1$. We will use this fact several times. In view of Lemma 4.2 and Lemma 4.1 we can assume that $d \geq 3$. Look at the graded free resolution of R/I :

$$\cdots \longrightarrow \bigoplus_i R(-\beta_{di}) \xrightarrow{\varphi_d} \cdots \longrightarrow \bigoplus_i R(-\beta_{0i}) \xrightarrow{\varphi_0} 0.$$

Apply $F^n(-)$, this induces the following complex C_{\bullet} :

$$\cdots \longrightarrow \bigoplus_i R(-\beta_{di}q) \longrightarrow \cdots \longrightarrow \bigoplus_i R(-\beta_{(d-3)i}q) \xrightarrow{F^n(\varphi_{d-3})} 0.$$

We apply Lemma 4.4 for $i = d-3$ and for the complex C_{\bullet} . As R is Cohen-Macaulay, the condition Lemma 4.4(1) satisfied. In the light of Lemma 4.11 $\dim H_k(C_{\bullet}) \leq d-2$, i.e., the property Lemma 4.4(2) checked. For the simplicity, set $R^{\beta_i} := \bigoplus_j R(-\beta_{ij})$. Note that $C_{d-(d-3)} = R^{\beta_{(2d-3)-(d-3)}} = R^{\beta_d}$. By Lemma 4.4,

$$\text{end}(H_{\mathfrak{m}}^{d-3}(H_0(C_{\bullet}))) \leq q \max\{\beta_{di}\} + \text{end}(H_{\mathfrak{m}}^d(R)) \leq aq + c, \quad (\natural)$$

for some a and c not depending to q .

(i) Without loss of generality we may assume that $\dim R = 3$. Note that $H_0(C_{\bullet}) = R/I^{[p^n]}$. In view of (\natural) , we have $\text{end}(H_{\mathfrak{m}}^0(R/I^{[p^n]})) \leq aq + c$. Lemma 4.3 completes the argument.

(ii) In view of i), without loss of generality we may assume that $\dim R \geq 4$. First we deal with the case $\dim R = 4$. Note that

$$R^{\beta_2} \xrightarrow{\varphi_2} R^{\beta_1} \longrightarrow I \longrightarrow 0,$$

is a presenting sequence of I . As the tensor product is right exact,

$$R^{\beta_2} \xrightarrow{F^n(\varphi_2)} R^{\beta_1} \xrightarrow{\rho} F^n(I) \longrightarrow 0$$

is exact. Therefore, $H_0(C_{\bullet}) = F^n(I)$. In a similar vein, we have the exact sequence

$$F^n(I) \xrightarrow{\phi_2} R \xrightarrow{\phi_1} F^n(R/I) \longrightarrow 0$$

Note that $\ker(\phi_1) = I^{[p^n]}$. There is a natural surjection

$$\pi : F^n(I) \longrightarrow \text{im}(\phi_2) = \ker(\phi_1) \longrightarrow 0$$

Set $L := \ker(\pi)$. Let \mathfrak{p} be a nonsingular prime ideal. Due to Fact 2.6 and Lemma 4.14 we have the exact sequence

$$0 \longrightarrow L_{\mathfrak{p}} \longrightarrow F^n(I_{\mathfrak{p}}) \longrightarrow (I_{\mathfrak{p}})^{[p^n]} \longrightarrow 0.$$

In view of Lemma 4.15, we get $L_{\mathfrak{p}} = 0$. According to the regularity of $\text{Proj}(\frac{R}{I})^2$, we observe that $\dim(L) < 2$. Keep the Grothendieck's vanishing theorem in mind. The exact sequence

$$0 \longrightarrow L \longrightarrow F^n(I) \longrightarrow I^{[p^n]} \longrightarrow 0$$

induces the exact sequence

$$H_{\mathfrak{m}}^1(F^n(I)) \longrightarrow H_{\mathfrak{m}}^1(I^{[p^n]}) \longrightarrow H_{\mathfrak{m}}^2(L) = 0 \quad (*)$$

Incorporate this with (†) and Lemma 4.12 to conclude that $\text{end}(H_{\mathfrak{m}}^1(I^{[p^n]})) \leq aq + c$. Then, in view of Lemma 4.10, $H_{\mathfrak{m}}^0(R/I^{[p^n]}) \simeq H_{\mathfrak{m}}^1(I^{[p^n]})$. Thus, $\text{end}(H_{\mathfrak{m}}^0(R/I^{[p^n]})) \leq aq + c$, for some a and c not depending to q . It remains to apply Lemma 4.3 to conclude the claim. Thus we can assume that $d > 4$. By looking at the exact sequence

$$0 \longrightarrow \text{im } F^n(\varphi_{d-4}) \longrightarrow \bigoplus_i R(-\beta_{(d-3)i}q) \longrightarrow H_0(C_{\bullet}) \longrightarrow 0,$$

we get that

$$H_{\mathfrak{m}}^{d-3}(H_0(C_{\bullet})) \simeq H_{\mathfrak{m}}^{d-2}(\text{im } F^n(\varphi_{d-4})).$$

By the following graded exact sequence

$$0 \longrightarrow \text{im}(F^n(\varphi_{d-4})) \longrightarrow R^{\beta_{d-3}} \longrightarrow R^{\beta_{d-3}}/\text{im}(F^n(\varphi_{d-4})) \longrightarrow 0$$

we observe

$$H_{\mathfrak{m}}^{d-3}\left(\frac{R^{\beta_{d-3}}}{\text{im}(F^n(\varphi_{d-4}))}\right) \simeq H_{\mathfrak{m}}^{d-2}(\text{im}(F^n(\varphi_{d-4}))).$$

Keep in mind

$$\ker(F^n(\varphi_{d-3}))/\text{im}(F^n(\varphi_{d-4})) \simeq \text{Tor}_{d-3}^R(F^n(R), R/I).$$

The following exact sequence of graded modules with homogeneous morphisms

$$0 \rightarrow \frac{\ker(F^n(\varphi_{d-3}))}{\text{im}(F^n(\varphi_{d-4}))} \rightarrow \frac{R^{\beta_{d-3}}}{\text{im}(F^n(\varphi_{d-4}))} \rightarrow \frac{R^{\beta_{d-3}}}{\ker(F^n(\varphi_{d-3}))} \rightarrow 0$$

implies the following exact sequence of graded modules:

$$H_{\mathfrak{m}}^{d-3}\left(\frac{R^{\beta_{d-3}}}{\text{im}(F^n(\varphi_{d-4}))}\right) \rightarrow H_{\mathfrak{m}}^{d-3}\left(\frac{R^{\beta_{d-3}}}{\ker(F^n(\varphi_{d-3}))}\right) \rightarrow H_{\mathfrak{m}}^{d-2}\left(\text{Tor}_{d-3}^R(F^n(R), \frac{R}{I})\right).$$

As $d > 4$, we have $d - 2 > \dim(\text{Tor}_{d-3}^R(F^n(R), \frac{R}{I}))$. By Grothendieck's vanishing theorem,

$$H_{\mathfrak{m}}^{d-2}(\text{Tor}_{d-3}^R(F^n(R), \frac{R}{I})) = 0.$$

In particular, $\text{end}(H_{\mathfrak{m}}^{d-2}(\text{Tor}_{d-3}^R(F^n(R), \frac{R}{I}))) \leq O(p^n)$. So, in view of Lemma 4.12,

$$\text{end}\left(H_{\mathfrak{m}}^{d-3}\left(\frac{R^{\beta_{d-3}}}{\ker(F^n(\varphi_{d-3}))}\right)\right) \leq O(p^n) \quad (\dagger).$$

An easy case: $\dim R = 5$. Recall from part (i) that there is an exact sequence

$$R^{\beta_2} \xrightarrow{F^n(\varphi_2)} R^{\beta_1} \xrightarrow{\rho} F^n(I) \longrightarrow 0.$$

Thus,

$$\frac{R^{\beta_2}}{\ker(F^n(\varphi_2))} = \text{im}(F^n(\varphi_2)) = \ker(\rho) \quad (\star)$$

and

$$0 \longrightarrow \ker(\rho) \longrightarrow R^{\beta_1} \longrightarrow F^n(I) \longrightarrow 0.$$

This induces the exact sequence

$$0 = H_{\mathfrak{m}}^1(R^{\beta_1}) \rightarrow H_{\mathfrak{m}}^1(F^n(I)) \rightarrow H_{\mathfrak{m}}^2(\ker(\rho)) \rightarrow H_{\mathfrak{m}}^2(R^{\beta_1}) = 0.$$

Therefore,

$$\begin{aligned} \text{end}(H_{\mathfrak{m}}^0(R/I^{[p^n]})) &\stackrel{4.10}{=} \text{end}(H_{\mathfrak{m}}^1(I^{[p^n]})) \\ &\stackrel{(*)}{\leq} \text{end}(H_{\mathfrak{m}}^1(F^n(I))) \\ &= \text{end}(H_{\mathfrak{m}}^2(\ker(\rho))) \\ &\stackrel{(*)}{=} \text{end}(H_{\mathfrak{m}}^2(\frac{R^{\beta_2}}{\ker(F^n(\varphi_2))})) \\ &\stackrel{(\dagger)}{\leq} aq + c \quad (\ddagger) \end{aligned}$$

So, Lemma 4.3 completes the argument in this case.

Now assume $d > 5$. Our next task is to descent (\ddagger) . This is a repetition of the above argument. We do this for the convenience of the reader. Keep in mind that

$$\frac{R^{\beta_{d-3}}}{\ker(F^n(\varphi_{d-3}))} \simeq \text{im}(F^n(\varphi_{d-3})) \quad (*, *)$$

Look at the exact sequence

$$0 \rightarrow \text{im}(F^n(\varphi_{d-3})) \rightarrow R^{\beta_{d-4}} \rightarrow \frac{R^{\beta_{d-4}}}{\text{im}(F^n(\varphi_{d-3}))} \rightarrow 0.$$

This gives

$$0 \rightarrow H_{\mathfrak{m}}^{d-4} \left(\frac{R^{\beta_{d-4}}}{\text{im}(F^n(\varphi_{d-3}))} \right) \rightarrow H_{\mathfrak{m}}^{d-3}(\text{im}(F^n(\varphi_{d-3}))) \rightarrow 0,$$

and by the help of $(*, *)$ and (\ddagger) we observe that

$$\text{end} \left(H_{\mathfrak{m}}^{d-4} \left(\frac{R^{\beta_{d-4}}}{\text{im}(F^n(\varphi_{d-3}))} \right) \right) \leq O(p^n).$$

Again look at

$$0 \rightarrow \frac{\ker(F^n(\varphi_{d-4}))}{\text{im}(F^n(\varphi_{d-5}))} \rightarrow \frac{R^{\beta_{d-4}}}{\text{im}(F^n(\varphi_{d-5}))} \rightarrow \frac{R^{\beta_{d-4}}}{\ker(F^n(\varphi_{d-4}))} \rightarrow 0.$$

As $d > 5$, we have $d - 3 > \dim(\text{Tor}_{d-4}^R(F^n(R), \frac{R}{I}))$. By Grothendieck's vanishing theorem,

$$\text{end} \left(H_{\mathfrak{m}}^{d-3}(\text{Tor}_{d-4}^R(F^n(R), \frac{R}{I})) \right) \leq O(p^n).$$

Hence, $\text{end} \left(H_{\mathfrak{m}}^{d-4} \left(\frac{R^{\beta_{d-4}}}{\ker(F^n(\varphi_{d-4}))} \right) \right) \leq O(p^n)$ i.e., (\ddagger) descended. Doing inductively, we get

$$\text{end} \left(H_{\mathfrak{m}}^2 \left(\frac{R^{\beta_2}}{\ker(F^n(\varphi_2))} \right) \right) \leq O(p^n).$$

By the same vein as (\ddagger) , we have $\text{end}(H_{\mathfrak{m}}^0(R/I^{[p^n]})) \leq aq + c$ and again Lemma 4.3 yields the claim.

(iii) This is in Corollary 4.5. \square

One may extend the above proof by computing $\mathrm{Tor}_i^R(F^n(R), R/I)$. Here we give an example of a graded ring R with a graded ideal I such that $\dim(\mathrm{Tor}_i^R(F^n(R), R/I)) = \dim R/I = \dim R$ for all $i > 0$.

Example 4.16. Let $R := \mathbb{F}_p[X_1, X_2, \dots, X_{\ell+1}]/(X_1 X_2)$. We use lowercase letters here to elements in R . Set $I := (x_1)$. Look at the graded free resolution of R/I :

$$F_\bullet : \cdots \longrightarrow R(-2) \xrightarrow{x_2} R(-1) \xrightarrow{x_1} R \longrightarrow 0.$$

As F_\bullet is minimal, $\mathrm{p.dim}(I) = \infty$. Then $F^n(F_\bullet)$ is of the form

$$\cdots \longrightarrow R(-2p^n) \xrightarrow{x_2^{p^n}} R(-p^n) \xrightarrow{x_1^{p^n}} R \longrightarrow 0.$$

Thus, (up to a shifting by $p^n - 1$)

$$\mathrm{Tor}_i^R(F^n(R), M) = \begin{cases} \frac{(0:Rx_1^{p^n})}{(x_2^{p^n})} \simeq \frac{(x_2)}{(x_2^{p^n})} & \text{if } i \in 2\mathbb{N}_0 + 1 \\ \frac{(0:Rx_2^{p^n})}{(x_1^{p^n})} \simeq \frac{(x_1)}{(x_1^{p^n})} & \text{if } i \in 2\mathbb{N} \end{cases}$$

and so

$$\mathrm{Ann}_R(\mathrm{Tor}_i^R(F^n(R), R/I)) = \begin{cases} (x_2^{p^n-1}) & \text{if } i \in 2\mathbb{N}_0 + 1 \\ (x_1^{p^n-1}) & \text{if } i \in 2\mathbb{N}. \end{cases}$$

Therefore, we can not control $\dim(\mathrm{Tor}_i^R(F^n(R), R/I))$ by some things independent of $\dim(R/I)$ for all $i > 0$ and all ℓ .

5. AN APPLICATION TO HILBERT-KUNZ MULTIPLICITY

Discussion 5.1. In view of [49, Page 3] we borrow the following quotation: "If one could check that (LC) holds for the Frobenius powers of the ideal I obtained in the first step of Brenner's construction, our result could then be used to obtain a more direct and explicit route to \mathfrak{m} -primary ideals with irrational Hilbert-Kunz multiplicities."

The ring in Brenner's construction is $R := K[X, Y, Z, W]/(F)$ where F is homogeneous of degree four and K has positive characteristic $p \gg 0$. The ideal in Brenner's construction is $I := (a, b)$ for some homogenous elements a and b in R .

Corollary 5.2. *Adopt the above notation. Then the (LC) holds for the proposed ring R .*

Proof. Note that R is normal, Cohen-Macaulay, $\dim R = 3$ and I is homogeneous. So the desired property follows from Theorem 1.2. \square

6. APPLICATIONS TO $e_{gHK}(-)$ OF 2-DIMENSIONAL RINGS

Among other things we give a proof of Corollary 1.4 and Corollary 1.5.

Fact 6.1. Let R be a ring of prime characteristic p , $I \triangleleft R$ an ideal and M a finitely generated module. The following holds.

- i) $\min(I) = \min(I^{[q]})$ for any $q := p^n$.
- ii) $\mathrm{Supp}(M) = \mathrm{Supp}(F^n(M))$ for all n .

Lemma 6.2. *Let R be a standard graded ring over a field of prime characteristic p and let M be a 1-dimensional finitely generated graded R -module. Then $\cup_n \mathrm{Ass}(F^n(M))$ is finite.*

Proof. Denote the unique maximal graded ideal of R by \mathfrak{m} and take

$$\mathfrak{p} \in \bigcup_n \text{Ass}(F^n(M)) \setminus \{\mathfrak{m}\}.$$

Since $\dim(M) = 1$ and in view of the above fact, we deduce

$$\mathfrak{p} \in \bigcup_n \min\{\text{Ass}_R(F^n(M))\} = \bigcup_n \min\{\text{Supp}(F^n(M))\} = \min\{\text{Supp}(M)\}.$$

Thus,

$$\bigcup_n \text{Ass}(F^n(M)) \subset \min\{\text{Supp}(M)\} \cup \{\mathfrak{m}\}$$

which is a finite set. \square

Remark 6.3. i) Suppose R is a semilocal ring and M is a finitely generated module. If $\dim(M) = 1$, then $\bigcup \text{Ass}(F^n(M))$ is finite. Indeed, apply the above argument.

ii) It may be $|\bigcup \text{Ass}(F^n(M))| = \infty$, see [30]. However, countable prime avoidance holds for rings that contain an uncountable field and holds in any complete local ring, see [12].

Lemma 6.4. (See [4] and [5]) *Let R be a two-dimensional normal standard graded K -domain over an algebraically closed field K of prime characteristic p and I a homogeneous ideal of dimension zero. Then $f_{HK}^{R/I}(q) = e_{HK}(R/I)q^2 + \gamma(q)$, where $e_{HK}(R/I)$ is a rational number and $\gamma(q)$ is a bounded function. Moreover if K is the algebraic closure of a finite field, then $\gamma(q)$ is an eventually periodic function.*

Discussion 6.5. Here is a comment on the construction of s in Lemma 3.6(ii). Set $\text{Ass}(F^n(R/I))^\circ := \text{Ass}(F^n(R/I)) \setminus \{\mathfrak{m}\}$. Due to [49, Proposition 2.4], $s \in \mathfrak{m}^{b(I)} \setminus \bigcup_n \text{Ass}(F^n(R/I))^\circ$. One can pick s to be homogeneous, if I and R are homogeneous.

Proof of Corollary 1.4. If $\dim R/I = 0$, then I is primary to the maximal ideal. In this case the generalized Hilbert-Kunz theory is the classical Hilbert-Kunz theory. The claim in this case is the subject of Lemma 6.4. If $\dim R/I = 2$, then $I = 0$ and $H_{\mathfrak{m}}^0(R/I^{[q]}) = 0$, so there are nothing to prove. Then, without loss of the generality we can assume that $\dim R/I = 1$. Due to Lemma 6.2, $\bigcup_n \text{Ass}(R/I^{[q]})$ is finite. In the light of Lemma 4.2(1), the (LC) condition holds. By Discussion 6.5, there is a homogeneous s such that

$$f_{gHK}^{R/I} = 2f_{gHK}^{R/I+(s)} - f_{gHK}^{R/I+(s^2)} \quad (\dagger)$$

Thanks to Lemma 6.4,

- 1) $f_{gHK}^{R/I+(s)}(n) = e'q^2 + \gamma'(q)$, where $e' \in \mathbb{Q}$ and $\gamma'(q)$ is bounded,
- 2) $f_{gHK}^{R/I+(s^2)}(n) = e''q^2 + \gamma''(q)$, where $e'' \in \mathbb{Q}$ and $\gamma''(q)$ is bounded,
- 3) if $K = \overline{\mathbb{F}}_p$, then $\gamma'(q)$ and $\gamma''(q)$ are eventually periodic.

Combining 1) and 2) along with (\dagger) yields that

$$\begin{aligned} f_{gHK}^{R/I} &= 2(e'q^2 + \gamma'(q)) - (e''q^2 + \gamma''(q)) \\ &= (2e' - e'')q^2 + (2\gamma'(q) - \gamma''(q)) \\ &:= e_{gHK}(R/I)q^2 + \gamma(q). \end{aligned}$$

So, *i)* follows. The item 3) presents the proof of *ii)*. \square

Lemma 6.6. ([38, Theorem 3.7]) *Let R be the coordinate ring of a nodal plane cubic projective curve over an algebraically closed field of prime characteristic p . Let J be a homogeneous primary to the irrelevant ideal. Then $f_{gHK}^{R/J}(n) = \mu q^2 + aq - r$, r only depends on $q \pmod{3}$. There is an explicit formula for r .*

Proof of Corollary 1.5. Without loss of the generality we assume $\text{ht}(I) = 1$. By Lemma 4.2(2), R satisfies in the (LC) condition. Lemma 6.2 says $\bigcup_n \text{Ass}(R/I^{[q]})$ is finite. In view of Lemma 3.6,

$$f_{gHK}^{R/I} = 2f_{HK}^{R/I+(s)} - f_{HK}^{R/I+(s^2)} \quad (\dagger)$$

for a homogeneous element s . In the light of Lemma 6.6,

- i) $f_{gHK}^{R/I+(s)}(q) = \mu'q^2 + a'q - r'$, where r is an integer that depends on $q \bmod 3$.
- ii) $f_{gHK}^{R/I+(s^2)}(q) = \bar{\mu}q^2 + \bar{a}q - \bar{r}$, where \bar{r} is an integer that depends on $q \bmod 3$.

Combining (1) and (2) throughout (\dagger) we get the claim. \square

Theorem 6.7. *Let I be a two generated graded ideal of the coordinate ring of an irreducible plane projective curve over an algebraically closed field of prime characteristic p . Then $e_{gHK}(R/I)$ is rational.*

Proof. Keep [5, Corollary 3.7] and Lemma 4.2(ii) in mind. Now the claim follows by the proof of Corollary 1.4. \square

In the above result our data on $f_{gHK}(R/I)$ becomes complete, if one can prove the primary version of the following result of Monsky.

Lemma 6.8. ([39, Theorem I and II]) *Let k be an algebraically closed field of characteristic $p > 0$, and $f \in k[x, y, z]$ be a degree d irreducible form defining a projective plane curve. Then*

$$f_{HK}(n) = e_{HK}(R/\mathfrak{m})p^{2n} + R(n)$$

where $R(n) = O(p^n)$. Suppose in addition k is finite. One has

- i) If $e_{HK}(R/\mathfrak{m}) = \frac{3}{4}d$, then $R(n)$ is eventually periodic.
- ii) If $e_{HK}(R/\mathfrak{m}) \neq \frac{3}{4}d$, then the $O(1)$ term for $R(n)$ is eventually periodic.

Toward extending the primary version of the mentioned result we restate the following question of Monsky asked in a workshop.

Question 6.9. Let C be a reduced irreducible projective curve over a finite field of characteristic p , and W be a vector bundle on C . Let $W^{[q]}$ be the pull-back of W by the n -th power of Frobenius. How does the element $\text{Poincare}(W^{[q]})$ of $\mathbb{Z}[T, 1/T]$ depend on n ?

Remark 6.10. Let R be as Corollary 1.4, and $f, g \in R$ be a homogeneous parameter sequence. Then

$$\ell(R/(f^n, g^m)) = mn\ell(R/(f, g))$$

for any n and m .

Proof. In the polynomial ring we have $\ell(\frac{K[X, Y]}{(X^n, Y^m)}) = mn$. Any normal and 2-dimensional ring is Cohen-Macaulay. Thus f, g is a regular sequence. In view of [21], there is a flat ring homomorphism $K[X, Y] \rightarrow R$ defined by the assignments $X \rightarrow f$ and $Y \rightarrow g$. By flatness, we have the following formula:

$$\begin{aligned} \ell(R/(f^n, g^m)) &= \ell\left(\left(\frac{K[X, Y]}{(X^n, Y^m)}\right) \otimes R\right) \\ &= \ell\left(\frac{K[X, Y]}{(X^n, Y^m)}\right) \ell\left(\frac{R}{(X, Y)R}\right) \\ &= \ell\left(\frac{K[X, Y]}{(X^n, Y^m)}\right) \ell\left(\frac{R}{(f, g)}\right) \\ &= mn\ell(R/(f, g)). \end{aligned}$$

\square

Corollary 6.11. (Also, see [7, Example 3.4]) Let R be as Corollary 1.4 and $f \in R$ be homogeneous. Then $f_{gHK}^{R/(f)} = 0$. In particular, $e_{gHK}(R/(f)) = 0$.

Proof. Set $I = (f)$ and let s be as Lemma 3.6. So,

$$\begin{aligned} f_{gHK}^{R/I}(n) &= 2f_{HK}^{R/I+(s)}(n) - f_{HK}^{R/I+(s^2)}(n) \\ &= 2\ell(R/(f^{p^n}, s^{p^n})) - \ell(R/(f^{p^n}, s^{2p^n})) \\ &= 0. \end{aligned}$$

□

In a similar vein we observe:

Corollary 6.12. Let A be a d -dimensional Cohen-Macaulay ring which contains a field and let x_1, \dots, x_d be a parameter sequence. Then $\ell(A/(x_1^{n_1}, \dots, x_d^{n_d})) = n_1 \dots n_d \ell(A/(x_1, \dots, x_d))$.

7. APPLICATIONS TO THE STABILITY

In this section we give a proof of Corollary 1.7 and Example 1.8. We need the following discussion in this and in the next section.

Discussion 7.1. Let \mathcal{V} be a vector bundle over a projective scheme X . Let $\{f_1, \dots, f_n\}$ be homogeneous elements of an \mathbb{N} -graded ring R . Set $d_i := \deg f_i$. The sheaf of relations $\mathcal{S} := \text{Syz}(f_1, \dots, f_n)$ on $\text{Proj}(R)$ is given by the following exact sequence:

$$0 \longrightarrow \mathcal{S} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_X(-d_i) \xrightarrow{f_1, \dots, f_n} \mathcal{O}_X.$$

If X is integral, then \mathcal{S} is a torsion-free sheaf, and \mathcal{S} is a vector bundle on $\text{Proj}(R)$ provided $\text{rad}(f_1, \dots, f_n) = \mathfrak{m}$. The slope of V is defined by $\mu(\mathcal{V}) := \frac{\deg(\mathcal{V})}{\text{rank}(\mathcal{V})}$. For example, $\text{rank}(\mathcal{S}) = n - 1$ and if the zero locus $Z = V(f_i)_{i=1}^n$ has codimension ≥ 2 , then

$$\deg(\mathcal{S}(m)) = \left((n-1)m - \sum_{i=1}^n d_i \right) \deg(X).$$

The sheaf \mathcal{V} is said to be *semistable* if $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$ for every subsheaf $\mathcal{W} \subset \mathcal{V}$ (subbundle when we deal with projective curves). If strict inequality holds for all such \mathcal{W} , we say \mathcal{V} is *stable*. A deep result of Donaldson-Uhlenbeck-Yau corresponds stable vector bundles over a complex manifold to Einstein-Hermitian vector bundles. This is quite strong. Any vector bundle V has a Harder-Narasimhan filtration, i.e., a chain

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \dots \subset \mathcal{V}_t = \mathcal{V}$$

such that $\frac{\mathcal{V}_i}{\mathcal{V}_{i-1}}$ is semistable and $\mu(\frac{\mathcal{V}_i}{\mathcal{V}_{i-1}}) > \mu(\frac{\mathcal{V}_{i+1}}{\mathcal{V}_i})$. We denote $\mu_{\min} := \mu(\frac{\mathcal{V}_1}{\mathcal{V}_0})$ and $\mu_{\max} := \mu(\mathcal{V}_t)$. Let $F : X \rightarrow X$ be the absolute Frobenius map. A vector bundle is called *strongly semistable* if all the Frobenius pull backs are again semistable. We recall that

$$\overline{\mu}_{\min}(\mathcal{V}) := \inf\{\mu_{\min}(F^{n*}(\mathcal{V})) / q\}$$

is a well-defined rational number by a result of Langer [36].

We will utilize the following result:

Lemma 7.2. ([5, Corollary 4.4]) *Let R be a two-dimensional normal standard-graded domain over an algebraically closed field K of positive characteristic p and let $Y = \text{Proj } R$ denote the corresponding smooth projective curve of genus g . Let $I = (f_1, \dots, f_3)$ denote an R_+ -primary homogeneous ideal generated by homogeneous elements. If $\text{Syz}(I)$ is strongly semistable, then*

$$e_{HK}(I) = \frac{\deg(Y)}{2} \left(\frac{(\deg(f_1) + \deg(f_2) + \deg(f_3))^2}{2} - (\deg(f_1)^2 + \deg(f_2)^2 + \deg(f_3)^2) \right).$$

Proof of Corollary 1.7. First note that s does not belong to any associated prime ideal of $R/I^{[q]}$ except for the maximal ideal, because $\text{Ass}(R/I^{[q]}) \subset \{\mathfrak{m}, I\}$. The same property holds for s^2 . Let (a, b, c) be the degree of (f, g, s) . In view of Lemma 3.6 and Discussion 6.5,

$$f_{gHK}^{R/I} = 2f_{HK}^{R/I+(s)} - f_{HK}^{R/I+(s^2)}.$$

Let $Y := \text{Proj}(R)$. If $\text{Syz}(f, g, s)$ and $\text{Syz}(f, g, s^2)$ are strongly semistable, then in view of Lemma 7.2

$$\begin{aligned} e_{gHK}(R/(f, g)) &= 2e_{HK}(R/I + (s)) - e_{HK}(R/I + (s^2)) \\ &= 2 \frac{\deg(Y)}{2} \left(\frac{(a+b+c)^2}{2} - (a^2 + b^2 + c^2) \right) \\ &\quad - \frac{\deg(Y)}{2} \left(\frac{(a+b+2c)^2}{2} - (a^2 + b^2 + 4c^2) \right) \\ &= \frac{\deg(Y)}{2} (-a^2 - b^2 - c^2 + 2ab + 2ac + 2bc) \\ &\quad + \frac{\deg(Y)}{2} (a^2/2 + b^2/2 + 2c^2 - ab - 2ac - 2bc) \\ &= \frac{\deg(Y)}{2} (-a^2/2 - b^2/2 + c^2 + ab) \\ &= \frac{\deg(Y)}{2} \left(c^2 - \frac{(a-b)^2}{2} \right) \quad (*). \end{aligned}$$

We now work with s^2 (resp. s^4) instead of s (resp. s^2). If both of

$$\{\text{Syz}(f, g, s^2), \text{Syz}(f, g, s^4)\}$$

are strongly semistable, then the similar computation says

$$e_{gHK}(R/(f, g)) = \frac{\deg(Y)}{2} \left(4c^2 - \frac{(a-b)^2}{2} \right) \quad (*).$$

Clearly, $(*) \neq (*)$, because $c \neq 0$. This is the contradiction. \square

Let \mathcal{C} be a degree four plane curve. Brenner proved that $\text{Syz}_{\mathcal{C}}(x^2, y^2, z^2)$ is semistable, see [9, Lemma 7.1] and he posted the following questions:

Question 7.3. ([9, Example 7.6]) Let $R := K[x, y, z]/(zx^3 + xy^3 + yz^3)$ and let $\mathcal{C} := \text{Proj}(R)$ be the corresponding curve. i) Is $\text{Syz}_{\mathcal{C}}(x^2, y^2, z^2)(3)$ strongly semistable in positive characteristic? ii) Is $\text{Syz}_{\mathcal{C}}(x^2, y^2, z^2)(3)$ stable?

Lemma 7.4. *Let R be as above and assume $p := \text{char}(R) = 2$. Set $\mu(i) := e_{HK}((x^i, y^i, z^i), R)$. Then $\mu(i) = 3i^2 + \frac{49}{16}(\delta^*(2t/7, 2t/7, 2t/7))^2$, where δ^* is the Han's operator. In particular,*

$$e_{HK}((x^2, y^2, z^2), R) = \mu(2) = p^2 \mu(1) = 4(3 + \frac{1}{24}).$$

Proof. The first assertion is in [41, Theorem 1.2]. The second assertion is in [41, Remark 1.3]. \square

Lemma 7.5. (See [29, Corollary 4.3.1]) *Let \mathcal{V} be a semistable vector bundle of rank 2 on smooth projective curve of genus $g \geq 2$ over an algebraically closed field of characteristic two. If $F^*\mathcal{V}$ is not semi-stable, then \mathcal{V} is stable.*

Fact 7.6. Let \mathcal{V} be a vector bundle. Then $\deg(F^*(\mathcal{V})) = p \deg(\mathcal{V})$. In particular, $\deg(\mathcal{V}) = 0$ implies $\deg(F^*(\mathcal{V})) = 0$.

One (non)-natural way to prove the fact over a curve \mathcal{C} is to identify \mathcal{V} with $\text{Syz}_{\mathcal{C}}(\underline{f})(m)$ and use $F^*(\text{Syz}_{\mathcal{C}}(\underline{f})(m)) \simeq \text{Syz}_{\mathcal{C}}(\underline{f}^p)(pm)$. Then the formula presented in Discussion 7.1 completes the proof.

Lemma 7.7. *Let R be a two-dimensional normal standard-graded domain over an algebraically closed field K of positive characteristic p and let $Y = \text{Proj } R$ denote the corresponding smooth projective curve. Then Frobenius pull-back is exact over quasi-coherent sheaves.*

Proof. Recall that normal rings satisfy in the Serre's condition $\mathcal{R}(1)$. In view of Fact 2.6 and Fact 2.5, the Frobenius map is flat over $\text{Proj}(R)$. In particular, the pull-back via Frobenius is exact as a functor over quasi-coherent sheaves. \square

Proof of Example 1.8. First note that the corresponding curve is nonsingular and projective, see [21, Page 305, 2.4]. Recall that $g := \dim H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is the genus. By [21, III) Ex. 4.7], $g = 3$. Set $\mathcal{V} := \text{Syz}_{\mathcal{C}}(x^2, y^2, z^2)(3)$.

i) Suppose on the contrary \mathcal{V} is strongly semistable and look for a contradiction. As, the semistability is independent of the shifting, $\text{Syz}(x^2, y^2, z^2)$ is strongly semistable. Then, in view of Lemma 7.2,

$$e_{HK}((x^2, y^2, z^2), R) = 12$$

which is not possible as Lemma 7.4 says.

ii) Recall from [9, Lemma 7.1] that \mathcal{V} is semistable. Now we show that it is stable. In view of part i) there is $n \in \mathbb{N}_0$ such that $F^{n*}(\mathcal{V})$ is semistable but $F^{(n+1)*}(\mathcal{V})$ is not semistable. This syzygy bundle is of rank two and of degree zero. Due to Lemma 7.5, $F^{n*}(\mathcal{V})$ is stable. If \mathcal{V} were not be stable it should have a line subbundle \mathcal{L} of degree zero. In the light of Lemma 7.7, $F^{n*}(\mathcal{L}) \subset F^{n*}(\mathcal{V})$. By Fact 7.6, it is of zero degree. This contradicts the stability of $F^{n*}(\mathcal{V})$. So, \mathcal{V} is stable. \square

The following characteristic-free realization may be helpful.

Remark 7.8. Having the above notation in mind. Due to [9, Example 7.6] there is a line bundle \mathcal{L} of degree -1 and an exact sequence ξ of the form

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{V} \longrightarrow \mathcal{L}^{-1} \longrightarrow 0.$$

As \mathcal{V} is semistable, ξ does not split. One may observe

$$\xi \in \text{Ext}^1(\mathcal{L}^{-1}, \mathcal{L}) \simeq H^1(\mathcal{L}^2) \simeq H^0(\omega_{\mathcal{C}} \otimes \mathcal{L}^{-2})^v$$

The base-point-free linear system $H^0(\omega \otimes \mathcal{L}^{-2})$ is of dimension $g + 1 = 4$. It defines an embedding $\varphi : \mathcal{C} \rightarrow \mathbb{P}^3$. We view ξ as an element in $\mathbb{P}(H^1(\mathcal{C}, \mathcal{L}^2)) \simeq \mathbb{P}^3$. Then by the proof of [18, Theorem 4.10],

$$\mathcal{V} \text{ is stable} \iff \xi \in \mathbb{P}^3 \setminus \varphi(\mathcal{C}).$$

8. F -THRESHOLD AND PROOF OF EXAMPLE 1.9

Rings in this section are not necessarily graded. Let I, J be two ideals with $J \subset \text{rad}(I)$. Set $v(q) := \sup\{\ell : J^\ell \not\subseteq I^{[q]}\}$. Look at the quantities $c_+^I(J) := \limsup_{n \rightarrow \infty} v(q)/q$ and $c_-^I(J) := \liminf_{n \rightarrow \infty} v(q)/q$. When these quantities are the same, we call it the F -threshold of I with respect to J denoted by $c^I(J)$. For example in the regular case and due to Fact 2.5, $\{v(q)/q\}$ is monotone and so $c_-^I(J) = c_+^I(J)$. Let us give an easy connection from $c(I)$ to F -threshold:

Observation 8.1. Let I be \mathfrak{m} -primary and suppose $c^I(\mathfrak{m})$ exists. Then $c(I) - 1 \leq c^I(\mathfrak{m}) \leq c(I)$.

Proof. By definition of $c(I)$, for each n there is $n' \geq n$ such that $\mathfrak{m}^{(c(I)-1)q'} \not\subseteq I^{[q']}$, where $q' := p^{n'}$. Thus, $v(q') \geq (c(I) - 1)q'$. To compute a limit its enough to deal with the subsequences, when the limit exists. So, $\lim_{n \rightarrow \infty} v(q)/q = \lim_{n' \rightarrow \infty} v(q')/q' \geq c(I) - 1$. Again by definition of $c(I)$, there is q_1 such that $\mathfrak{m}^{c(I)q} \subseteq I^{[q]}$ for all $q > q_1$. Thus, $v(q) \leq c(I)q - 1$, and so $\limsup_{n \rightarrow \infty} v(q)/q \leq c(I)$. \square

In general $c^I(\mathfrak{m})$ is not necessarily an integer. However, the bound may be sharp:

Example 8.2. Let (R, \mathfrak{m}) be a d -dimensional regular local ring of prime characteristic. Then $c(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m}) = d$.

Proof. As R is regular, \mathfrak{m} is generated by a full system of parameters. Thus the equality $c^{\mathfrak{m}}(\mathfrak{m}) = d$ is in [27, Example 2.7]. The other equality follows by the proof of [27, Example 2.7]. Let us present the short argument. As \mathfrak{m} is generated by d elements, $\mathfrak{m}^{dq} \subseteq \mathfrak{m}^{[q]}$. So, $c(\mathfrak{m}) \leq d$. Suppose on the contradiction that $\mathfrak{m}^{dq-q} \subseteq \mathfrak{m}^{[q]}$ for all $q \gg 0$. Take $q > d$ be large enough. Then $dq - d > dq - q$. Hence, $\mathfrak{m}^{dq-d} \subset \mathfrak{m}^{dq-q} \subseteq \mathfrak{m}^{[q]}$. This contradicts the monomial conjecture which is a theorem in this situation. \square

Recall that \overline{I} the integral closure of an ideal I is the set of all elements $r \in R$ that there exist $a_i \in I^i$ such that $r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$. The ring in Example 1.9 is complete-intersection and of low-dimension.

Proof of Example 1.9. Let $R := \mathbb{F}_p[[t^2, t^3]]$, $\mathfrak{m} := (t^2, t^3)$ and $J := (t^2)$. Then R is a Cohen-Macaulay local ring of characteristic $p > 0$ with $d = \dim(R) = 1$ and J is generated by a full system of parameters. Set $f(X) := X^2 - t^6 \in R[X]$. Note that $t^6 = t^2 \cdot t^4 \in J \cdot J = J^2$. Then $f(t^3) = 0$. So $t^3 \in \overline{J}$, i.e., $\mathfrak{m} = (t^2, t^3) \subseteq \overline{J} \subseteq \mathfrak{m}$. Recall that $\mathfrak{m}^2 = (t^4, t^5, t^6) \subseteq J$. Thus,

$$a := \max\{n \mid \mathfrak{m}^n \not\subseteq J\} = 1.$$

Recall that [27, Question 3.5] claims $\mathfrak{m}^s \subseteq \overline{J}$ if and only if $s \geq \frac{a}{d} + 1$. The only possible case for s is the case $s = 1$, i.e., $\mathfrak{m} \subseteq \overline{J}$. If the discussed question were be the case then we should have $s = 1 \geq 2 = \frac{a}{d} + 1$ which is a contradiction. \square

9. CONNECTING TO THE WALDSCHMIDT CONSTANT

Let R be a \mathbb{N} -graded integral domain and $I \triangleleft R$ be homogeneous. Set $\alpha(I) := \min\{n : I_n \neq 0\}$.

Definition 9.1. The Waldschmidt constant of I is defined by $\gamma(I) := \lim_{n \rightarrow \infty} \frac{\alpha(I^{(n)})}{n}$.

The next result is well-known over polynomial rings [2]:

Lemma 9.2. *The above limit exists.*

Proof. As $I^{(n)} I^{(m)} \subset I^{(n+m)}$, we have $\alpha(I^{(n+m)}) \leq \alpha(I^{(n)}) + \alpha(I^{(m)})$. Due to the Feketes lemma, we get the desired limit. \square

Corollary 9.3. (Also, see [47]) *There is c such that $\mathfrak{m}^{cn} \mathfrak{p}^{(n)} \subset \mathfrak{p}^n$, where \mathfrak{p} is a prime homogeneous ideal of dimension one in a standard graded ring (R, \mathfrak{m}) satisfying the Serre's condition $\mathcal{S}(2)$ over a field of any characteristic.*

Proof. Recall that $\text{end}(H_{\mathfrak{m}}^1(\mathfrak{p}^n)) \leq an + b$ for $n \gg 0$, see e.g. [16]. By the same reason as Lemma 4.10, $\text{end}(H_{\mathfrak{m}}^0(R/\mathfrak{p}^n)) \leq an + b$ for $n \gg 0$. As, $\dim R/\mathfrak{p} = 1$ and from the fact $\mathfrak{p}^{(n)}$ is a \mathfrak{p} -primary component of \mathfrak{p}^n , one may observe that $(\mathfrak{p}^n)^{\text{sat}} = \mathfrak{p}^{(n)}$. By the same reason as Lemma 4.3, there is c such that

$$0 = \mathfrak{m}^{cn} H_{\mathfrak{m}}^0(R/\mathfrak{p}^n) = \mathfrak{m}^{cn} \frac{(\mathfrak{p}^n)^{\text{sat}}}{\mathfrak{p}^n} = \mathfrak{m}^{cn} \frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^n},$$

which yields the claim. \square

Proposition 9.4. *Let \mathfrak{p} be a prime homogeneous ideal of dimension one in a standard graded ring (R, \mathfrak{m}) over a field of any characteristic. Then $\gamma(\mathfrak{p}) \geq \alpha(\mathfrak{p}) - d(\mathfrak{p})$.*

Proof. Recall from Definition 1.6 that $0 = \mathfrak{m}^{d(\mathfrak{p})n} H_{\mathfrak{m}}^0(R/\mathfrak{p}^n) = \mathfrak{m}^{d(\mathfrak{p})n} \frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^n}$ for all $n \gg 0$. Thus $\mathfrak{m}^{d(\mathfrak{p})n} \mathfrak{p}^{(n)} \subset \mathfrak{p}^n$ for all $n \gg 0$. By the degree containment, $\alpha(\mathfrak{p}^{(n)}) + d(\mathfrak{p})n \geq \alpha(\mathfrak{p}^n) = n\alpha(\mathfrak{p})$ for all $n \gg 0$. To compute a limit, it's enough to deal with the tail of the sequence. Taking the limit, we get the desired claim. \square

The above bound is sharp:

Example 9.5. Let $R := k[x, y]$ and let $\mathfrak{p} := (x)$. Due to $\mathfrak{p}^n = \mathfrak{p}^{(n)} = (x^n)$, we have $\gamma(\mathfrak{p}) = \alpha(\mathfrak{p}) = 1$. As $R/\mathfrak{p}^n = k[x, y]/(x^n)$ is Cohen-Macaulay and of dimension one, $H_{\mathfrak{m}}^0(R/\mathfrak{p}^n) = 0$. Then $d(\mathfrak{p}) = 0$. So, $\gamma(\mathfrak{p}) = \alpha(\mathfrak{p}) - d(\mathfrak{p})$.

Let us give more examples:

Example 9.6. Let k be a field and $R := k[X, Y, Z]/(Z^2 - XY)$. The ideal $\mathfrak{p} := (x, z)$ is homogeneous, prime and of dimension one. The following holds:

- i) $\gamma(\mathfrak{p}) = 1/2$,
- ii) $d(\mathfrak{p}) = 1$,
- iii) $\lim_{n \rightarrow \infty} \ell(\frac{H_{\mathfrak{m}}^0(R/\mathfrak{p}^n)}{n^{\dim R}}) = 1/4$.

Proof. First we compute the symbolic powers. Let $x^i y^j z^k \in \mathfrak{m}^{2n} \cap (x^n)$. Hence $i \geq n$ and $i + j + k = 2n$. Thus $j \leq i$. If $j = 0$, then $x^i y^j z^k \in \mathfrak{p}^{2n}$. In the case $j > 0$ and due to $z^2 = xy$ we have $x^i y^j z^k = x^{i-j} (x^j y^j z^k) = x^{i-j} z^{k+2j}$. This belongs to \mathfrak{p}^{2n} . Thus, $(x^n) \cap \mathfrak{m}^{2n} \subseteq \mathfrak{p}^{2n}$. The other side inclusion deduces in a similar way. So,

$$\mathfrak{p}^{2n} = (x^n) \cap \mathfrak{m}^{2n}.$$

Now we deal with odd integers. Note that

$$\begin{aligned} \mathfrak{p}^{2n+1} &= (x^{2n+1}, \dots, x^{n+1} z^n, x^n z^{n+1}, \dots, z^{2n} x, z^{2n+1}) \\ &= (x^{2n+1}, \dots, x^{n+1} z^n, x^{n+1} y z^{n-1}, \dots, y^n x^{n+1}, z^{2n+1}) \\ &\subseteq (x^{n+1}, z^{2n+1}). \end{aligned}$$

This yields $\mathfrak{p}^{2n+1} \subseteq (x^{n+1}, z^{2n+1}) \cap \mathfrak{m}^{2n+1}$. Reversely, suppose $x^i y^j z^k \in (x^{n+1}, z^{2n+1}) \cap \mathfrak{m}^{2n+1}$ where $k = 0$ or $k = 1$. Look at the following possibilities:

- a) The case $k = 1$: We have $i + j = 2n$. As $z^{2n+1} = z x^n y^n$, $(x^{n+1}, z^{2n+1}) \subset (x^n)$. Conclude that $i \geq n$. Hence $j \leq n$. Thus $x^i y^j z^k = x^{i-j} z^{2j+1} \in \mathfrak{p}^{2n+1}$.
- b) The case $k = 0$: We have $i + j = 2n + 1$. As $z^{2n+1} = z x^n y^n$, $i \geq n$. Hence $j \leq n + 1$. We claim that $j \neq n + 1$. Suppose on the contrary that $x^n y^{n+1} \in (x^{n+1}, z^{2n+1}) \cap \mathfrak{m}^{2n+1}$. This shows $x^n y^{n+1} \in \mathfrak{p}^{(2n+1)}$. As $H_{\mathfrak{m}}^0(R/\mathfrak{p}^{2n+1}) \simeq \mathfrak{p}^{(2n+1)}/\mathfrak{p}^{2n+1}$, $x^n y^{n+1} + \mathfrak{p}^{2n+1}$ is annihilated by

some power of y . There is $\ell > 0$ such that $x^n y^{n+1+\ell} \in \mathfrak{p}^{2n+1}$ which is impossible. Thus $j \leq n$.

Therefor, $x^i y^j z^k = x^{i-j} z^{2j} \in \mathfrak{p}^{2n+1}$.

So, $\mathfrak{p}^{2n+1} = (x^{n+1}, z^{2n+1}) \cap \mathfrak{m}^{2n+1}$. Quickly, we deduce

$$\mathfrak{p}^{(i)} = \begin{cases} (x)^{\frac{i}{2}} & \text{if } i \in 2\mathbb{N} \\ (x^{\frac{i+1}{2}}, z^i) & \text{if } i \notin 2\mathbb{N} \end{cases} \implies \alpha(\mathfrak{p}^{(i)}) = \begin{cases} \frac{i}{2} & \text{if } i \in 2\mathbb{N} \\ \frac{i+1}{2} & \text{if } i \notin 2\mathbb{N}. \end{cases}$$

Now we are in a position to prove the claims:

i) We note that

$$\gamma(\mathfrak{p}) = \lim_{n \rightarrow \infty} \frac{\alpha(\mathfrak{p}^{(n)})}{n} = \lim_{n \rightarrow \infty} \frac{\alpha(\mathfrak{p}^{(2n)})}{2n} = \lim_{n \rightarrow \infty} \frac{n}{2n} = 1/2.$$

ii) As \mathfrak{p} is of dimension one, $H_{\mathfrak{m}}^0(R/\mathfrak{p}^i) \simeq \frac{\mathfrak{p}^{(i)}}{\mathfrak{p}^i}$. By the above computation, $H_{\mathfrak{m}}^0(R/\mathfrak{p}^{2n}) \simeq \frac{(x^n)}{\mathfrak{p}^{2n}}$ and $H_{\mathfrak{m}}^0(R/\mathfrak{p}^{2n+1}) \simeq \frac{(x^{n+1}, z^{2n+1})}{\mathfrak{p}^{2n+1}}$. We claim that $\mathfrak{m}^n H_{\mathfrak{m}}^0(R/\mathfrak{p}^n) = 0$. We do this for even n . The other case follows in a same way. Let i, j be such that $i + j + 1 = 2n$. Suppose first that $j \leq n$. Then $i \geq n - 1$ and $i + n + 1 \geq 2n$. Hence, $x^{n+i} y^j z \in (x, z)^{2n}$. Now assume $j > n$. We get that

$$x^{n+i} y^j z = x^i y^{n-j} z^{1+2n} \in (x, z)^{2n}.$$

In both cases $zx^i y^j \in \text{Ann}(H_{\mathfrak{m}}^0(R/\mathfrak{p}^{2n}))$. In a similar vein, $x^i y^j \in \text{Ann}(H_{\mathfrak{m}}^0(R/\mathfrak{p}^{2n}))$ with $i + j = 2n$. So $\mathfrak{m}^{2n} H_{\mathfrak{m}}^0(R/\mathfrak{p}^{2n}) = 0$.

iii) For any $0 \leq i < n$, define $A_i := \{x^{n+i} y^k : 0 \leq k < n - i\}$. For any $0 \leq j \leq n - 2$, define $B_j := \{x^{n+j} y^\ell z : 1 \leq \ell \leq n - 1 - j\}$. Finally, set $C := \{x^{n+j} z : 0 \leq j \leq n - 2\}$. Let $x^i y^j z^k \in (x^n) \setminus \mathfrak{p}^{2n}$. Set $\Gamma := (\bigcup_\ell A_\ell) \cup (\bigcup_m B_m) \cup C$. Due to $z^2 = xy$, we observe $x^i y^j z^k \in \Gamma$. Also, Γ consists of K -linearly independent elements. Thus,

$$\begin{aligned} \ell(H_{\mathfrak{m}}^0(R/\mathfrak{p}^{2n})) &= \sum |A_i| + \sum |B_j| + |C| \\ &= \sum_{i=0}^{n-1} |n - i| + \sum_{j=0}^{n-2} |n - 1 - j| + (n - 1) \\ &= \frac{(n+1)n}{2} + \frac{n(n-1)}{2} + n - 1 \\ &= n^2 + n - 1. \end{aligned}$$

So,

$$\lim_{n \rightarrow \infty} \ell\left(\frac{H_{\mathfrak{m}}^0(R/\mathfrak{p}^n)}{n^2}\right) = \lim_{n \rightarrow \infty} \ell\left(\frac{H_{\mathfrak{m}}^0(R/\mathfrak{p}^{2n})}{(2n)^2}\right) = 1/4.$$

□

Remark 9.7. i) On $\mathbb{P}_{\mathbb{C}}^N$ and over any fat ideal I , Chudnovsky's conjecture says

$$\frac{\alpha(I) + \dim \mathbb{P}_{\mathbb{C}}^N - 1}{\dim \mathbb{P}_{\mathbb{C}}^N} \leq \frac{\alpha(I^{(n)})}{n}.$$

By the above example this is not true for general projective schemes.

ii) The inequality is true for any homogeneous ideal consisting a linear form. Indeed, we have $\alpha(I) = 1$. By Euler's formula,

$$I^n \subseteq I^{(n)} \subseteq \mathfrak{m} I^{(n-1)} \subseteq \mathfrak{m}^{n-1} I.$$

Read this as

$$n = n - 1 + \alpha(I) = \alpha(\mathfrak{m}^{n-1} I) \leq \alpha(I^{(n)}) \leq n\alpha(I) = n.$$

Hence, $\alpha(I^{(n)}) = n$ and the claim follows.

iii) Moreover, if I is prime and consisting a 2-form the inequality holds for $n = 2$. Indeed, $3 = \alpha(\mathfrak{m} I) \leq \alpha(I^{(2)}) \leq 4 = 2\alpha(I)$. In the case $\alpha(I^{(2)}) = 4$ there is nothing to prove. Also, $N = 1$ implies that

$\text{ht}(I) = 1$. It turns out that I is principal. We conclude by this that $I^{(2)} = I^2$ and so $\alpha(I^{(2)}) = 4$. Then, without loss of generality may assume that $\alpha(I^{(2)}) = 3$ and $N \geq 2$. It remains to show $3/2 \geq \frac{N+1}{N}$. This always holds, as $N \geq 2$.

10. REMARKS ON A QUESTION OF HERZOG

Recall the following from [23]:

Question 10.1. Let (R, \mathfrak{m}) be a 3-dimensional regular local ring and \mathfrak{p} a prime ideal of dimension one. What is $\ell(\mathfrak{p}^{(n)}/\mathfrak{p}^n)$?

There is a simple algorithm to deal with this:

Corollary 10.2. *Let R be a standard graded algebra over a field and \mathfrak{p} a prime ideal of dimension one. Then there is a homogenous x such that $\ell(\mathfrak{p}^{(n)}/\mathfrak{p}^n) = 2\ell(\frac{R}{\mathfrak{p}^n + (x^n)}) - \ell(\frac{R}{\mathfrak{p}^n + (x^{2n})})$.*

Proof. Look at the graded family $\{\mathfrak{p}^m\}$. Having the proof of Corollary 9.3, there is a such that $\mathfrak{m}^{aq}H_{\mathfrak{m}}^0(R/\mathfrak{p}^n) = 0$. The set $\Delta := \{\mathfrak{p} : \mathfrak{p} \in \text{Ass}(R/\mathfrak{p}^n)\}$ is finite. In fact $\Delta \subset \min(\mathfrak{p}) \cup \{\mathfrak{m}\}$. Now, it is enough to take $x \in \mathfrak{m}^a \setminus \mathfrak{p}$ and apply Lemma 3.6. \square

Corollary 10.3. *Adopt the above notation. Then $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ if and only if*

$$\ell\left(\frac{R}{\mathfrak{p}^n + (x^{2n})}\right) = 2\ell\left(\frac{R}{\mathfrak{p}^n + (x^n)}\right).$$

We state a Frobenius version of Question 10.1.

Proposition 10.4. *Let $I \triangleleft R := \mathbb{F}_p[X_1, \dots, X_m]$. The following holds:*

- i) $f_{g_{HK}}^{R/I}(n) = e_{g_{HK}}(R/I)q^n$.
- ii) $e_{g_{HK}}(R/I)$ realizes as a length of a module. In particular, $e_{g_{HK}}(R/I) \in \mathbb{N}_0$.
- iii) $e_{g_{HK}}(R/I) > 0$ if and only if $\dim(R/I) = \dim R$.

Proof. By graded local duality [11, Theorem 3.6.19], $H_{\mathfrak{m}}^0(R/I^{[q]}) \cong \text{Hom}(\text{Ext}_R^m(R/I^{[q]}, R), \mathbb{E})$. Look at the free resolution of R/I :

$$0 \longrightarrow R^\ell \xrightarrow{A} R^{\ell'} \longrightarrow \dots \longrightarrow R^{\mu(I)} \longrightarrow R.$$

By $(-)^t$ we mean the transpose of a matrix $(-)$. Let $B_n := F^n(A)^t$. By Fact 2.5, $F^n(-)$ is exact. Hence, the free resolution of $R/I^{[q]}$ is

$$0 \longrightarrow R^\ell \xrightarrow{B_n} R^{\ell'} \longrightarrow \dots \longrightarrow R^{\mu(I)} \longrightarrow R.$$

Thus,

$$\begin{aligned} \text{Ext}_R^m(R/I^{[q]}, R) &= \text{coker}(R^{\ell'} \xrightarrow{B_n} R^\ell) \\ &\simeq F^n(\text{coker}(R^{\ell'} \xrightarrow{A^t} R^\ell)), \end{aligned}$$

the last one follows by the exactness of $F^n(-)$. Due to [43, Thm. 2], one has

$$\begin{aligned} \ell(H_{\mathfrak{m}}^0(F^n(R/I))) &= \ell\left(F^n(\text{coker}(R^{\ell'} \xrightarrow{A^t} R^\ell))\right) \\ &= p^{mn}\ell\left(\text{coker}(R^{\ell'} \xrightarrow{A^t} R^\ell)\right). \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{\ell(H_{R_+}^0(R/I^{[p^n]}))}{p^{mn}} = \ell\left(\text{coker}(R^{\ell'} \xrightarrow{A^t} R^\ell)\right) \in \mathbb{N}_0.$$

Claims follows by this. \square

Example 10.5. Let $R_p := \mathbb{F}_p[x_0, \dots, x_3]$ and $I_p := (x_0^2, x_1^2, x_0x_2 + x_1x_3)$. The following holds.

- i) $\lim_{p \rightarrow \infty} \frac{\ell(H_{\mathfrak{m}_p}^0(R_p/I_p^{[p^n]}))}{p^{n \dim R_p}} = 1.$
- ii) $\lim_{p \rightarrow \infty} \frac{\ell(H_{\mathfrak{m}_p}^0(R_p/I_p^{[p^n]}))}{p^{n \dim R_p}}$ does not depend on n .
- iii) $\lim_{p \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \frac{\ell(H_{R_+}^0(R_p/I_p^{[p^n]}))}{p^{n \dim R_p}} \right) = 1.$

Proof. When there is no confusion, we drop the index p and we use R (resp. I and \mathfrak{m}) instead of R_p (resp. I_p and \mathfrak{m}_p). Set

$$A_1 := \begin{pmatrix} -x_1^2 & x_0^2 & 0 \\ 0 & -x_0x_2 & x_1^2 \\ -x_0x_2 - x_1x_3 & 0 & x_0^2 \\ -x_1x_2 & -x_0x_3 & x_0x_1 \\ -x_2^2 & x_3^2 & x_0x_2 - x_1x_3 \end{pmatrix}^t$$

and

$$A_2 := \begin{pmatrix} x_2 & x_3 & 0 & 0 \\ x_0 & 0 & 0 & x_3 \\ 0 & -x_1 & -x_2 & 0 \\ -x_1 & x_0 & x_3 & x_2 \\ 0 & 0 & x_3 & x_1 \end{pmatrix}$$

Then, in view of [42, Example 2.4], the free resolution of R/I is of the form

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -x_3 \\ x_2 \\ -x_1 \\ x_0 \end{pmatrix}} R^4 \xrightarrow{A_2} R^5 \xrightarrow{A_1} R^3 \xrightarrow{\begin{pmatrix} x_0^2 \\ x_1^2 \\ x_0x_2 + x_1x_3 \end{pmatrix}^t} R.$$

By Proposition 10.4,

$$\begin{aligned} \ell(H_{\mathfrak{m}}^0(F^n(R/I))) &= \ell(R/(x_0^{p^n}, \dots, x_3^{p^n})) \\ &= p^{4n}. \end{aligned}$$

The claims follows by this. □

Remark 10.6. i) Adopt the above notation. Then $H_{\mathfrak{m}}^1(\bigoplus I^{[p^n]}/I^{[p^{n+1}]}) \neq 0$. Indeed, note that I is a 3-generated ideal in a 4-dimensional ring. So, $\dim R/I \neq 0$. The claim follows by the following item (note that the completion of R is an integral domain and local cohomology behave nicely with completion).

ii) Let (R, \mathfrak{m}) be a complete local and Cohen-Macaulay domain. Let I be an ideal such that $H_{\mathfrak{m}}^1(\bigoplus I^{[p^n]}/I^{[p^{n+1}]}) = 0$. Then $f_{g_{HK}}^{R/I}(n) \neq 0$ (up to a subsequence) if and only if $\dim R/I = 0$. Indeed,

if $\dim R/I = 0$ this is clear that $f_{gHK}^{R/I}(n) = f_{HK}^{R/I}(n) \neq 0$. Up to a subsequence, we have

$$\begin{aligned}
 f_{gHK}^{R/I}(n) \neq 0 &\implies H_{\mathfrak{m}}^0(R/I^{[p^n]}) \neq 0 \\
 &\xrightarrow{1} \varprojlim H_{\mathfrak{m}}^0(R/I^{[p^n]}) \neq 0 \\
 &\xrightarrow{2} (\varprojlim H_{\mathfrak{m}}^0(R/I^{[p^n]}))^v \neq 0 \\
 &\xrightarrow{3} \varinjlim \operatorname{Ext}_R^d(R/I^{[p^n]}, \omega_R) \neq 0 \\
 &\xrightarrow{4} \operatorname{cd}(I, \omega_R) = d \\
 &\xrightarrow{5} \operatorname{cd}(I, R) = d \\
 &\xrightarrow{6} \dim R/I = 0,
 \end{aligned}$$

where:

- 1) The assumption shows $H_{\mathfrak{m}}^0(R/I^{[p^{n+1}]}) \longrightarrow H_{\mathfrak{m}}^0(R/I^{[p^n]})$ is surjective. The inverse limit of nonzero modules with surjective maps between them is nonzero.
- 2) Set $(-)^v := \operatorname{Hom}(-, E(R/\mathfrak{m}))$. This is exact as a contravariant functor.
- 3) Any Cohen-Macaulay ring which is quotient of a regular ring has a canonical module ω_R . Then local duality works, see [11, Theorem 3.5.8].
- 4) Denote $\sup\{i : H_I^i(M) \neq 0\}$ by $\operatorname{cd}(I, M)$.
- 5) We remark that $\operatorname{cd}(I, R) = \sup\{\operatorname{cd}(I, M)\} \leq \dim R$.
- 6) Lichtenbaum-Hartshorne vanishing theorem (over a complete local domain) says that $\dim R/I = 0$ if and only if $\operatorname{cd}(I, R) = \dim R$.

11. MORES ON NUMERICAL INVARIANTS

Lemma 11.1. *Let $R := \frac{\mathbb{F}_p[X, Y, Z]}{(X^3 + Y^3 + Z^3)}$ be the Fermat cubic. Then*

$$\mathfrak{m}^{2q+1} H_{\mathfrak{m}}^0(R/I^{[p^n]}) = 0 \neq \mathfrak{m}^{2q} H_{\mathfrak{m}}^0(R/I^{[p^n]}).$$

Proof. First we show $z^2 x^{q-1} y^{q-1} \notin \operatorname{Ann}_R H_{\mathfrak{m}}^0(R/I^{[p^n]}) = R/I^{[p^n]}$. Suppose on the contrary that there are E, F and $G \in \mathbb{F}_p[X, Y, Z]$ such that

$$Z^2 X^{q-1} Y^{q-1} + F(X^3 + Y^3 + Z^3) = EX^q + GY^q \quad (*)$$

By looking at the monomial terms of $\{E, F, G\}$ and applying the monomial grading on both sides of $(*)$ there are the following three possibilities:

- 1) $Z^2 X^{q-1} Y^{q-1} + F_1 X^3 = E_1 X^q + G_1 Y^q$, or
- 2) $Z^2 X^{q-1} Y^{q-1} + F_1 Y^3 = E_1 X^q + G_1 Y^q$, or
- 3) $Z^2 X^{q-1} Y^{q-1} + F_1 Z^3 = E_1 X^q + G_1 Y^q$,

where $\{E_1, F_1, G_1\}$ are the monomial terms of $\{E, F, G\}$. Here we adopt zero as a monomial. The third one occurs, if its both sides are zero. As $Z^3 \nmid Z^2 X^{q-1} Y^{q-1}$, this is not the case. Suppose 2) holds. This implies the following equalities:

- i) $F_1 X^3 = E_2 X^q + G_2 Y^q$, and
- ii) $F_1 Z^3 = E_3 X^q + G_3 Y^q$, and
- iii) $F_1 = -Z^2 X^{q-1} Y^{q-4}$,

where $\{E_i, G_i\}$ are the monomial terms of $\{E, G\}$. Putting iii) along with ii), implies that

$$Z^2 X^{q-1} Y^{q-4} = E_3 X^q + G_3 Y^q.$$

This equation has no solution in the polynomial ring and by this we get a contradiction that we search for it. So, 2) is not the case. Similarly, 1) yields a contradiction. In sum,

$$z^2 x^{q-1} y^{q-1} \notin \text{Ann}_R H_{\mathfrak{m}}^0(R/I^{[p^n]}).$$

To show $\mathfrak{m}^{2q+1} \subseteq I^{[p^n]}$, without loss of the generality let $f := x^i y^j z^k \in \mathfrak{m}^{2q+1}$. Due to the relation $x^3 + y^3 + z^3 = 0$ and by working with terms separately, we may assume that $k < 3$. If $i \geq q$ we get $f \in (x^q, y^q) = I^{[q]}$ and the claim follows. Then we can assume that $i < q$. As $k < 3$ we get from $i + j + k = 2q + 1$ that $j \geq q$ and so $f \in (x^q, y^q) = I^{[q]}$. In particular,

$$\mathfrak{m}^{2q+1} H_{\mathfrak{m}}^0(R/I^{[p^n]}) = \mathfrak{m}^{2q+1} \left(\frac{R}{I^{[p^n]}} \right) = 0.$$

□

Remark 11.2. If $\mathfrak{m}^{cq} H_{\mathfrak{m}}^0(F^n(R/I)) = 0$ for all $q \gg 0$ it follows for all q after possible enlarging of c . Clearly, $c(I) \leq b(I)$. This may be strict, as the next result says.

Example 11.3. Let $R := \frac{\mathbb{F}_2[X, Y, Z]}{(X^3 + Y^3 + Z^3)}$ be the Fermat cubic and $J := (x^2, y^2)$. Then

$$c(J) = 3 < 4 = b(J).$$

Proof. Let $I := (x, y)$. Then $J^{[q]} = I^{[q+1]}$ and in view of Lemma 11.1

$$\mathfrak{m}^{2q+3} H_{\mathfrak{m}}^0(R/J^{[q]}) = 0 \neq \mathfrak{m}^{2q+2} H_{\mathfrak{m}}^0(R/J^{[q]}) \quad (*).$$

Then for all $q \geq 2^2$,

$$\mathfrak{m}^{3q} H_{\mathfrak{m}}^0(R/J^{[p^n]}) = 0 \neq \mathfrak{m}^{2q} H_{\mathfrak{m}}^0(R/J^{[p^n]}) \quad (*, *).$$

Thus, $c(J) = 3$. Now we apply $(*)$ in the case $q = 2$. This yields

$$\mathfrak{m}^{4p} H_{\mathfrak{m}}^0(R/J^{[p]}) = 0 \neq \mathfrak{m}^{3p} H_{\mathfrak{m}}^0(R/J^{[p]}).$$

This along with $(*, *)$ shows $b(J) = 4$. □

We continue by connecting the invariants to the tight closure theory. Recall that the Frobenius closure of an ideal I is $I^F := \{x \in R : x^q \in I^{[q]} \quad \exists q > 0\}$.

Discussion 11.4. Let I be an \mathfrak{m} -primary ideal of an standard graded algebra R over a field of prime characteristic. The following holds:

i) Let $c := c(I)$. Then $R_{\geq c} \subseteq I^F$.

ii) If $\mathfrak{m}^{\alpha q + \beta} H_{\mathfrak{m}}^0(F^n(R/I)) = 0$ for some α and β that do not depending to q , then $R_{\geq \alpha} \subset I^*$.

Proof. i) Look at $1 + I^{[p^n]} \in R/I^{[p^n]} = H_{\mathfrak{m}}^0(R/I^{[p^n]})$, where \mathfrak{m} is the homogeneous maximal ideal. Multiplying by \mathfrak{m}^{cq} , we deduce $\mathfrak{m}^{cq} \subset I^{[q]}$. Let $x \in R_{\geq c} = R_1^c R$. Then $x = \sum y_i^c x_i$ where $y_i \in R_1$ and $x_i \in R$. Thus $x^q = \sum y_i^{cq} x_i^q \in \mathfrak{m}^{cq} \subset I^{[q]}$, i.e., $x \in I^F$.

ii) Without loss of the generality we may assume that $\dim R \neq 0$. We take $x \in R_{\geq \alpha} = R_1^\alpha R$ and look at $\mathfrak{m}^{\alpha q + \beta} \subset I^{[q]}$. Fix $c \in \mathfrak{m}^\beta$ which is not in the union of the minimal prime ideals. Then $x = \sum y_i^\alpha x_i$ where $y_i \in R_1$ and $x_i \in R$. So, $y_i^{q\alpha} x_i^q \in \mathfrak{m}^{\alpha q}$ and $cx^q = \sum c y_i^{q\alpha} x_i^q \in \mathfrak{m}^\beta \mathfrak{m}^{\alpha q} \subset I^{[q]}$, i.e., $x \in I^*$. □

Despite of triviality of the (LC) property when I is \mathfrak{m} -primary, computing $b(I)$ is not so easy for us. We apply a powerful computational method of Brenner [6] to find upper bounds on $b(I)$. The bound depends on degree data coming from the ideal and the ring.

Remark 11.5. Let C be a smooth plane curve defined by the equation $f = 0$. Denote the coordinate ring of C by R . There is an uniform bound on the (LC)-exponent of two-generated ideal $(g, h)R$ by fixing the degree of g and h . If $\text{ht}(g, h) = 2$, the bound depends only on $\{\deg f, \deg g, \deg h\}$.

Proof. Over smooth curves torsion-free sheaves are vector bundle. Without loss of the generality we may assume that I is not principal. As $I := (f, g)$ is two generated the corresponding syzygy bundle is a line bundle. Denote it by $\text{Syz}(I)$ and set $(d, d_1, d_2) := (\deg f, \deg g, \deg h)$. Set $e := \frac{\deg(\tilde{I})}{d}$ which is independent of q and look at the exact sequence of torsion-free sheaves:

$$0 \longrightarrow \text{Syz}(I) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}(-d_i) \xrightarrow{f, g} \tilde{I} \longrightarrow 0.$$

Then $\deg(\text{Syz}(I)) = -(d_1 + d_2 + e)d$. We note that $e = 0$ when I is primary to the maximal ideal. Every line bundle is strongly semistable. Thus

$$\bar{\mu}_{\min}(\text{Syz}(I)) = \mu_{\min}(\text{Syz}(I)) = \deg(\text{Syz}(I)) = -(d_1 + d_2 + e)d.$$

Denote the homogeneous maximal ideal by \mathfrak{m} . Recall by Lemma 4.10 that $H_{\mathfrak{m}}^0(R/I^{[p^n]})_m \simeq H_{\mathfrak{m}}^1(I^{[p^n]})_m$. Then, by the same lines as [6] and for all

$$\begin{aligned} m &> -q \frac{\bar{\mu}_{\min}(\text{Syz}(I))}{\deg C} + \frac{\deg(\omega)}{\deg C} + 1 \\ &= q(d_1 + d_2 + e) + \frac{(d-1)(d-2)-2}{d} + 1 \end{aligned}$$

we have $H_{\mathfrak{m}}^0(R/I^{[p^n]})_m = 0$. Set

$$\begin{aligned} \alpha &:= d_1 + d_2 + e, \text{ and} \\ \beta &:= \frac{(d-1)(d-2)-2}{d} + 1. \end{aligned}$$

Now Lemma 4.3 yields $\mathfrak{m}^{\alpha q + \beta} H_{\mathfrak{m}}^0(R/I^{[p^n]}) = 0$. □

Discussion 11.6. If R is \mathbb{Z} -algebra and I is an ideal, there is a natural way to study the (LC) exponent by focusing on the asymptotic behavior of the sequence $\{b(I_p) : p \text{ is prime}\}$ when R_p and I_p are reduction mod p .

Example 11.7. i) Let $R := \frac{\mathbb{F}_p[X, Y, Z]}{(X^3 + Y^3 + Z^3)}$ be the Fermat cubic. We use lowercase letters here to elements in R . Look at $I := (x, y)$. Denote the homogeneous maximal ideal by \mathfrak{m} . In view of Lemma 11.1, $b(I) = c(I) = 3$. This is well-known that $z^2 \in I^* \setminus I^F$. In particular, the bounds presented by Discussion 11.4 and Remark 11.5 achieved. However, as Example 11.12 declares, the bound may be strict, even we deal with \mathbb{P}^1 .

ii) Let R be as i). Look at $I := xR$. As R is Cohen-Macaulay and x^q is a parameter element, we get that

$$\inf\{i : H_{\mathfrak{m}}^i(R/I^{[p^n]}) \neq 0\} = \text{depth}(R/x^q R) = \text{depth } R - 1 = 1.$$

So, $b(I) = 0$.

iii) Let $R := \frac{\mathbb{Z}[X, Y, Z]}{(XY - Z^2)}$ and $I := (x, y)$. We use R_p and I_p to reduction mod p . Denote the homogeneous maximal ideal by \mathfrak{m}_p . Here, we compute $\lim_{p \rightarrow \infty} b(I_p)$. We claim that

$$\mathfrak{m}_p^{2q} H_{\mathfrak{m}_p}^0(R_p/I_p^{[p^n]}) = 0 \neq \mathfrak{m}_p^{2q-1} H_{\mathfrak{m}_p}^0(R_p/I_p^{[p^n]}).$$

By a same reasoning as presented in Lemma 11.1, we have $zx^{q-1}y^{q-1} \notin \text{Ann}_R H_{\mathfrak{m}}^0(R_p/I_p^{[p^n]})$. In order to check the left hand side, recall that $H_{\mathfrak{m}_p}^0(R_p/I_p^{[p^n]}) = R_p/I_p^{[p^n]}$ and let $f := x^i y^j z^k \in \mathfrak{m}_p^{2q}$. Due to the relation $z^2 = xy$ we assume that $k \in \{0, 1\}$. If $i \geq q$ we get $f \in (x^q, y^q)$ and the claim follows. Then we

can assume that $i < q$. As $k < 2$ we get from $i + j + k = 2q$ that $j \geq q$ and so $f \in (x^q, y^q)$. This proves the claim. So, $\lim_{p \rightarrow \infty} b(I_p) = 2$.

We need the following result of Herzog and Hibi.

Lemma 11.8. ([22, Theorem 4.1]) *Given a bounded increasing function $f : \mathbb{N} \rightarrow \mathbb{N}_0$. There exists a monomial ideal I in a polynomial ring R over any field such that $\text{depth}(R/I^k) = f(k)$ for all k .*

Example 11.9. Let I be a graded ideal in a polynomial ring R over a field of prime characteristic. It may be $\sup\{b(I^n) : n \in \mathbb{N}\} = \infty$.

The following argument works for the next result too.

Proof. Let n_0 be any positive integer. Look at

$$f(n) = \begin{cases} 0 & \text{if } n \leq n_0 \\ 1 & \text{if } n > n_0 \end{cases}$$

In view of Lemma 11.8, there is a homogeneous ideal I in a polynomial ring R such that $\text{depth}(R/I^n) = f(n)$. Suppose on the contrary that there is ℓ such that $\sup\{b(I^n)\}_{n=1}^\infty < \ell$. Set $\Gamma := \{\text{Ass}(R/I^n) : n \in \mathbb{N}\}$. By [10], we know $|\Gamma| < \infty$. As R is regular, $\text{Ass}(M) = \text{Ass}(F^n(M))$. In particular,

$$|\{\text{Ass}(R/F^m(I^n)) : n, m \in \mathbb{N}\}| = |\Gamma| < \infty.$$

In the light of Lemma 3.6 and Discussion 6.5, there is a uniform $s \in \mathfrak{m}^\ell \setminus \bigcup_{\mathfrak{p} \in \Gamma} \mathfrak{p}$ such that

$$f_{gHK}^{R/I^n} = 2f_{HK}^{R/I^n+(s)} - f_{HK}^{R/I^n+(s^2)}.$$

As R is regular, Hilbert-Kunz multiplicity should be colength. Set $R_1 := R/(s)$ (resp. $R_2 := R/(s^2)$) and $I_1 := IR_1$ (resp. $I_2 := IR_2$). Denote the Hilbert-Samuel polynomial of a graded A -module M by P_A^M . Let $n \gg 0$. These imply that

$$\begin{aligned} e_{gHK}(R/I^n) &= 2\ell\left(\frac{R}{I^n+(s)}\right) - \ell\left(\frac{R}{I^n+(s^2)}\right) \\ &= 2\ell\left(\frac{R_1}{I_1^n}\right) - \ell\left(\frac{R_2}{I_2^n}\right) \\ &= 2P_{R_1}^{I_1}(n) - P_{R_2}^{I_2}(n). \end{aligned}$$

Thus, $g(n) := e_{gHK}(R/I^n)$ is of polynomial type. Keep in mind any module over R is of finite projective dimension. By Auslander-Buchsbaum formula [11, Theorem 1.3.3],

$$\text{p. dim}(R/I^n) + \text{depth}(R/I^n) = \dim R.$$

In view of Proposition 10.4(iii)

$$g(n) = e_{gHK}(R/I^n) = 0 \iff f(n) = \text{depth}(R/I^n) \neq 0.$$

In particular, $g \neq 0$. As g is nonzero, the vanishing set of g is a finite set. So, the non-vanishing set of f is finite. This is a contradiction. \square

If I and J projectively have the same integral closure, then in view of [45], $e(R/IJ) = F(e(R/I), e(R/J))$ for some polynomial F , where $e(-)$ is the Hilbert-Samuel multiplicity.

Proof of Example 1.10. First recall that a closure operation is a map send an ideal I to another ideal denoted by I^c such that $I \subset I^c$, $I^c = (I^c)^c$ and the map is order-preserving. Let us recall that I and J

projectively have the same closure operation, if $(I^n)^c = (J^m)^c$ for some n and m in \mathbb{N} . Suppose on the contrary that there is a polynomial function F such that

$$F(e_{gHK}(R/I), e_{gHK}(R/J)) = e_{gHK}(R/IJ) \quad (*)$$

and look for a contradiction. Set

$$g(n) = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n > 1 \end{cases}$$

In view of Lemma 11.8, there is a homogeneous ideal I in the 3-dimensional ring $R := \mathbb{F}[X, Y, Z]$ such that $\text{depth}(R/I^n) = g(n)$. Let $J := I^{n-1}$ and look at

$$f(n) := F(e_{gHK}(R/I), e_{gHK}(R/I^{n-1})).$$

This is a polynomial in one variable. Again, we combine Auslander-Buchsbaum formula with Proposition 10.4(iii) to get

$$f(n) \stackrel{(*)}{=} e_{gHK}(R/I^n) = 0 \iff \text{depth}(R/I^n) \neq 0.$$

Note that $\text{depth}(R/I) = 0$, i.e., $f \neq 0$. As f is nonzero, the vanishing set of f is finite, but $\{n : \text{depth}(R/I^n) \neq 0\}$ is much more than a finite set. This is a contradiction that we search for it. \square

Fact 11.10. Let I be an ideal of a (graded) local ring (R, \mathfrak{m}) . Recall from Fact 2.2 that I^{sat} computed as the intersection of all primary to nonmaximal prime ideals of I . Let $x \in \mathfrak{m}$ but not in the other associated primes of $I^{[q]}$. Then $(I^{[q]} : x) \subset (I^{[q]})^{sat}$. Suppose now that

$$\mathfrak{m}^{\alpha q + \beta} H_{\mathfrak{m}}^0(R/I^{[q]}) = 0.$$

Let $c \in \mathfrak{m}^\beta$ and take $0 \neq x \in \mathfrak{m}^\alpha$ but not in the other associated primes of $I^{[q]}$. This implies that $(I^{[q]} : cx^q) = (I^{[q]})^{sat}$. Now assume $\emptyset \neq \bigcap \min((I^{[q]})^{sat})$, i.e. $\text{rad}(I) \neq \mathfrak{m}$, and let \mathfrak{q} be in. In this case one can pick $x \in \mathfrak{m}^\alpha \setminus I^*$. In particular, $1 \notin (I^{[q]} : cx^q)$. Then

$$\ell\left(\frac{R_{\mathfrak{q}}}{(I^{[q]} : cx^q)_{\mathfrak{q}}}\right) = \ell\left(\frac{R_{\mathfrak{q}}}{(I^{[q]})^{sat} R_{\mathfrak{q}}}\right) < \infty.$$

Motivated from [24] we ask:

Question 11.11. What is the asymptotic behavior of $\frac{\ell(R_{\mathfrak{q}}/(I^{[q]})^{sat} R_{\mathfrak{q}})}{q^{\dim R_{\mathfrak{q}}}}$?

Example 11.12. Let $R := \overline{\mathbb{F}}_p[X, Y]$ and let $I := (XY, X^n)$. The following holds:

- i) $c(I) = b(I) = n$. In particular, the bound given by Remark 11.5 may be strict.
- ii) $\lim_{q \rightarrow \infty} \frac{\ell(R_{\mathfrak{q}}/(I^{[q]})^{sat} R_{\mathfrak{q}})}{q^{\dim R_{\mathfrak{q}}}} = 1$.
- iii) $f_{gHK}^{R/I}(n) = (n-1)q^2$, and so $e_{gHK}(R/I) = n-1$.

Proof. First note that $I^{[q]} = (X^q Y^q, X^{nq})$. Its primary decomposition is given by

$$I^{[q]} = (X^q) \cap (Y^q, X^{nq}).$$

In view of Fact 2.2, $(I^{[q]})^{sat} = (X^q)$.

i) Denote the homogeneous maximal ideal by \mathfrak{m} . Recall that $H_{\mathfrak{m}}^0(R/I^{[q]}) = \frac{(I^{[q]})^{sat}}{I^{[q]}} = \frac{(X^q)}{(X^q Y^q, X^{nq})}$. One has $c(I) = b(I) = n$ provided:

$$\mathfrak{m}^{(n-1)q} H_{\mathfrak{m}}^0(R/I^{[q]}) \neq 0 = \mathfrak{m}^{nq} H_{\mathfrak{m}}^0(R/I^{[q]}).$$

Clearly, $(X^{(n-2)q-1}Y)X^q \in \mathfrak{m}^{(n-1)q} \setminus (X^qY^q, X^{nq})$. This clarifies $\mathfrak{m}^{(n-1)q}H_{\mathfrak{m}}^0(R/I^{[q]}) \neq 0$. In order to check the right hand side, without loss of the generality, we look at X^iY^j where $i+j = nq$. If $j < q$, then $i > (n-1)q$. Hence $(X^iY^j)X^q \in (X^qY^q, X^{nq})$. Thus we assume $j \geq q$ and this implies $(X^iY^j)X^q \in (X^qY^q, X^{nq})$. So, $c(I) = b(I) = n$.

Now we prove the particular case. As $\mathfrak{m} \in \text{Ass}(R/I)$, $\text{depth}(R/I) = 2$. Keep in mind $\text{p.dim}(R/I) < \infty$. By Auslander-Buchsbaum formula, $\text{p.dim}(R/I) = 2$. Its free resolution is the Taylor complex attached to XY and X^n . So, the following complex is exact:

$$0 \longrightarrow R(-n-1) \xrightarrow{\binom{-Y}{X^{n-1}}} R(-n) \oplus R(-2) \xrightarrow{(X^n, XY)} I \longrightarrow 0.$$

We get the following exact sequence of locally free sheaves over \mathbb{P}^1 :

$$0 \longrightarrow \mathcal{O}(-n-1) \longrightarrow \mathcal{O}(-n) \oplus \mathcal{O}(-2) \longrightarrow \tilde{I} \longrightarrow 0.$$

Therefore, $\deg(\tilde{I}) = -1$. In fact, by Grothendieck's theorem [20, Page 384, 2.6], one has $\tilde{I} = \mathcal{O}(-1)$. By the notation as Remark 11.5,

$$\begin{aligned} \alpha &:= d_1 + d_2 + e = n + 1, \text{ and} \\ \beta &:= \frac{(d-1)(d-2)-2}{d} + 1 = -1, \end{aligned}$$

we deduce

$$0 = \mathfrak{m}^{\alpha+\beta}H_{\mathfrak{m}}^0(R/I^{[q]}) = \mathfrak{m}^{(n+1)q-1}H_{\mathfrak{m}}^0(R/I^{[q]}).$$

From this we get $\mathfrak{m}^{(n+1)q}H_{\mathfrak{m}}^0(R/I^{[q]}) = 0$. So, the bound given by Remark 11.5 may be strict.

ii) Set $\mathfrak{q} := (X)$. Then $\{\mathfrak{q}\} = \bigcap \min((I^{[q]})^{sat})$. Note that $(A, \mathfrak{m}) := (R_{\mathfrak{q}}, \mathfrak{q}R_{\mathfrak{q}})$ is a regular local ring of dimension one. Therefore,

$$\lim_{n \rightarrow \infty} \frac{\ell(R_{\mathfrak{q}}/(I^{[q]})_{\mathfrak{q}}^{sat})}{q^{\dim R_{\mathfrak{q}}}} = \lim_{n \rightarrow \infty} \frac{\ell(A/\mathfrak{m}^{[p^n]})}{p^n} = e_{HK}(A) = 1.$$

iii) Define

$$\begin{aligned} A_1 &:= \{x^q y^j : 1 \leq j \leq q-1\}, \\ A_2 &:= \{x^{q+i} : 1 \leq i \leq (n-1)q-1\}, \text{ and} \\ A_3 &:= \{x^{q+i} y^j : 1 \leq i \leq (n-1)q-1, 1 \leq j \leq q-1\}. \end{aligned}$$

Then $\cup A_i$ spans $\frac{(X^q)}{(X^qY^q, X^{nq})}$ as a vector space, and its cardinality is

$$(q-1) + ((n-1)q-1) + ((n-1)q^2 + 2 - (n+1)q) = (n-1)q^2.$$

Now

$$f_{gHK}^{R/I}(n) = \ell\left(H_{\mathfrak{m}}^0(R/I^{[q]})\right) = \dim_{\mathbb{F}_p}\left(\frac{(X^q)}{(X^qY^q, X^{nq})}\right) = (n-1)q^2,$$

and the claim follows. \square

12. CONCLUDING REMARKS OVER LOCAL RINGS

We start by the following:

Discussion 12.1. Recall that a local ring R is of finite Cohen-Macaulay type, i.e., if there are, up to isomorphism, only a finite number of indecomposable maximal Cohen-Macaulay modules. By a celebrated result of Auslander, R has isolated singularity, provided it is Cohen-Macaulay.

Remark 12.2. Let (R, \mathfrak{m}) be a complete Cohen-Macaulay local ring and of finite Cohen-Macaulay type. Then (LC) holds over R .

The proof works in a more general setting: local rings of finite F -representation type.

Proof. Let $\underline{x} := x_1, \dots, x_d$ be a system of parameters. Then \underline{x} is a regular sequence on R . This implies $\underline{x}^q := x_1^q, \dots, x_d^q$ is a regular sequence. Therefore, \underline{x} is a regular sequence on $F^n(R)$. In particular, $F^n(R)$ is maximal Cohen-Macaulay module. As R is complete, the Krull-Remak-Schmidt holds for the category of finitely generated R -module. As R is F -finite, $F^n(R) = \bigoplus M_i(n)$ where $M_i(n)$ are finitely generated R -modules. Note that

$$\frac{M_i(n)}{IM_i(n)} \simeq M_i(n) \otimes R/I$$

has a right R -module structure coming from R/I , we denote this as

$$R \times (M_i(n)/IM_i(n)) \longrightarrow M_i(n)/IM_i(n)$$

and remark that $H_{\mathfrak{m}}^0(\frac{M_i(n)}{IM_i(n)})$ is finitely generated and \mathfrak{m} -torsion with respect to this multiplicative structure. In particular, there is $a_{i,n}$ such that

$$\mathfrak{m}^{a_{i,n}} \times H_{\mathfrak{m}}^0(\frac{M_i(n)}{IM_i(n)}) = 0.$$

Clearly, $M_i(n)$ are maximal Cohen-Macaulay. As R is of finite Cohen-Macaulay type, $\{M_i(n) : i, n\}$ is a finite set. Let $a := \max\{a_{i,n}\} < \infty$ and denote the minimal number of generators of \mathfrak{m}^a by ℓ . One may find easily that:

- 1) $\mathfrak{m}^{a\ell q} \subseteq (\mathfrak{m}^a)^{[q]}$,
- 2) $\mathfrak{m}^a \times H_{\mathfrak{m}}^0(F^n(R/I)) = 0$, and
- 3) $\mathfrak{m}^a \times F^n(R/I) = (\mathfrak{m}^a)^{[q]} \frac{R}{I^{[q]}}$.

Now $H_{\mathfrak{m}}^0(F^n(R/I)) = \bigoplus H_{\mathfrak{m}}^0(\frac{M_i(n)}{IM_i(n)})$. Set $b := a\ell$. Its enough to apply these items to observe

$$\mathfrak{m}^{bq} H_{\mathfrak{m}}^0(R/I^{[q]}) \subseteq (\mathfrak{m}^a)^{[q]} H_{\mathfrak{m}}^0(R/I^{[q]}) = \mathfrak{m}^a \times H_{\mathfrak{m}}^0(F^n(R/I)) = 0.$$

□

Let us recall the following from [1]:

Discussion 12.3. i) For a pair $N \subset M$ of R -modules, look at the action of Psekine-Szpiro on it, i.e., the map $F^n(N) \rightarrow F^n(M)$ and we write $N_M^{[q]} := \text{im}(F^n(N) \rightarrow F^n(M))$. Then N_M^* defined in a similar vein as above. Set $G^n(M) := F^n(M)/0_{F^n(M)}^*$. One has $G^n(R/\mathfrak{a}) = R/(\mathfrak{a}^{[q]})^*$.

ii) A complex $(G_{\bullet}, \varphi_{\bullet}) : 0 \rightarrow G_{\ell} \rightarrow \dots \rightarrow G_0 \rightarrow 0$ of finitely generated projective modules is called stably phantom acyclic if $\ker(F^e(\varphi_n)) \subset (\text{im}(F^e(\varphi_{n+1})))_{F^e(G^n)}^*$ for all n and all e . In this case we say $H_0(G_{\bullet})$ is of finite phantom projective dimension.

iii) (Phantom acyclicity criterion): Let R be a homomorphic image of a Cohen-Macaulay ring and G_{\bullet} be a bounded complex of finitely generated projective modules of constant rank. Then G_{\bullet} is stably phantom acyclic if and only if $G_{\bullet} \otimes \frac{R}{\text{rad}(0)}$ satisfies in the standard conditions on rank and midheight.

iv) If R is Cohen-Macaulay, then phantom projective dimension is the same as projective dimension.

Combining [1, Theorem 7.4] and [26] yields:

Remark 12.4. Let (R, \mathfrak{m}) be a local equidimensional ring satisfies in the phantom acyclicity criterion, e.g. homomorphic image of a Cohen-Macaulay ring, with an \mathfrak{m} -primary test ideal. Let I be an ideal of finite phantom projective dimension. Then (LC) holds for I .

Proof. In view of [1, Theorem 7.4], there is e such that

$$\mathfrak{m}^{e[q]} H_{\mathfrak{m}}^0(R/I^{[q]*}) = \mathfrak{m}^{e[q]} H_{\mathfrak{m}}^0(G^n(R/I)) = 0.$$

In particular, there is d such that $\mathfrak{m}^{dq} H_{\mathfrak{m}}^0(R/I^{[q]*}) = 0$. Denote the *test ideal* by τ . Let c be such that $\mathfrak{m}^c \subset \tau$. Thus,

$$\mathfrak{m}^c \left(\frac{I^{[q]*}}{I^{[q]}} \right) \subset \tau \left(\frac{I^{[q]*}}{I^{[q]}} \right) = 0.$$

Look at the exact sequence

$$0 \longrightarrow H_{\mathfrak{m}}^0 \left(\frac{I^{[q]*}}{I^{[q]}} \right) \longrightarrow H_{\mathfrak{m}}^0 \left(\frac{R}{I^{[q]}} \right) \longrightarrow H_{\mathfrak{m}}^0 \left(\frac{R}{I^{[q]*}} \right).$$

Set $b := c + d$. We deduce by this exact sequence that $\mathfrak{m}^{bq} H_{\mathfrak{m}}^0(R/I^{[q]}) = 0$ as claimed. \square

Corollary 12.5. *Let R be a weakly F -regular local ring and I be an ideal of finite projective dimension. Then (LC) holds for I .*

Proof. This is well-known that weakly F -regular local rings are Cohen-Macaulay, integral domain and their test ideal is the whole ring. Also, over Cohen-Macaulay rings phantom projective dimension is the same as projective dimension. This finishes the proof via the above argument. \square

In the regular case we find a bound for the (LC) exponent:

Remark 12.6. i) Let (R, \mathfrak{m}) be a d -dimensional regular local ring and M a finite length module. Then $\mathfrak{m}^{qd\ell(M)} F^n(M) = 0$. Indeed, we do induction by $\ell := \ell(M)$. If $\ell = 1$, then $M = R/\mathfrak{m}$ and the claim is clear in this case, because \mathfrak{m} is generated by d elements. Now suppose, inductively, that $\ell > 1$ and the result has been proved for modules of length less than ℓ . Look at the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow R/\mathfrak{m} \longrightarrow 0,$$

where $\ell(N) = \ell - 1$. By inductive hypothesis, $\mathfrak{m}^{qd(\ell-1)} F^n(N) = 0$. By Fact 2.5, there is an exact sequence

$$0 \longrightarrow F^n(N) \longrightarrow F^n(M) \xrightarrow{f} F^n(R/\mathfrak{m}) \longrightarrow 0.$$

Thus, $\mathfrak{m}^{dq} F^n(M) \subseteq \ker(f) \simeq F^n(N)$ and so

$$\mathfrak{m}^{qd\ell} F^n(M) = \mathfrak{m}^{qd(\ell-1)} \mathfrak{m}^{qd} F^n(M) \subseteq \mathfrak{m}^{qd(\ell-1)} F^n(N) = 0.$$

ii) Having the first item in mind and in order to find a sharp bound on $c(M)$, we deal with the case $M := R/I$ is cyclic. By [27, Example 2.7], $c^I(\mathfrak{m}) = \max\{r : \mathfrak{m}^r \not\subseteq I\} + d$. In view of Observation 8.1 one gets to a sharp bound.

iii) Let (R, \mathfrak{m}) be a regular local ring and M an R -module. Then $F^n(H_{\mathfrak{m}}^i(M)) \simeq H_{\mathfrak{m}}^i(F^n(M))$. Indeed, due to Fact 2.5, $F^n(-)$ is flat and a flat functor computes with the local cohomology modules.

iv) Let (R, \mathfrak{m}) be a d -dimensional regular local ring and M a finitely generated R -module. Then there is some b that does not depending to q such that $\mathfrak{m}^{bq} H_{\mathfrak{m}}^0(F^n(M)) = 0$. Indeed, set $b := \ell(H_{\mathfrak{m}}^0(M))d$. By the third item, $F^n(H_{\mathfrak{m}}^0(M)) \simeq H_{\mathfrak{m}}^0(F^n(M))$. In view of the first item, the claim is clear.

Remark 12.7. Let (R, \mathfrak{m}) be a regular local ring of prime characteristic and $I \triangleleft R$. In the same vein as Proposition 10.4 we have the following assertions:

- i) $f_{gHK}^{R/I}(n) = e_{gHK}(R/I)q^n$.
- ii) $e_{gHK}(R/I)$ realizes as a length of a module. In particular, $e_{gHK}(R/I) \in \mathbb{N}_0$.
- iii) $e_{gHK}(R/I) > 0$ if and only if $\text{p. dim}(R/I) = \dim R$.

Let us present a new argument for i) and ii). By Fact 2.5, $F^n(-)$ is flat. Any flat functor computes the local cohomology modules. Thus $F^n(H_m^0(M)) \simeq H_m^0(F^n(M))$. In particular, $f_{gHK}^{R/I}(n) = p^{n \dim R} \ell(H_m^0(R/I))$.

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